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Applications of finite arithmetic, I[1]

ROY DUBISCH, *Fresno State College, Fresno, California.*

Finite arithmetics may be used to considerable advantage in problems involving the factorization of polynomials with integral coefficients.

IN A FINITE ARITHMETIC we perform computations with the integers as we do in ordinary arithmetic except that we consider two integers to be equal if their difference is divisible by some fixed integer m. For example, if $m=5$ we may consider the five numbers of our finite arithmetic to be 0, 1, 2, 3, and 4 since every integer has a remainder of 0, 1, 2, 3, or 4 when divided by 5. Then, for example, $2+3=0$ since $5-0$ is divisible by 5; $3+4=2$ since $7-2$ is divisible by 5; $2\times3=1$ since $6-1$ is divisible by 5; $3\times4=2$ since $12-2$ is divisible by 5; $(-3)\times4=3=-2$ since $-12-3$ and $3-(-2)$ are both divisible by 5. We will call such a system a finite arithmetic modulo m and will denote it by A_m.

The above brief account of such systems is given mainly to refresh the memory of the reader, since extensive descriptions of finite arithmetics have been given in numerous readily available books and articles.[2]

What is lacking in the descriptions referred to, however, is any indication of possible applications of finite arithmetics to topics in elementary algebra. What we will show here is that finite arithmetics, especially A_2 and A_3, may be used to considerable advantage in problems involving the factorization of polynomials with integral coefficients.

The reasons for this are quite simple. First, we note that whenever we have the equality of two integers in ordinary arithmetic we certainly have equality in any finite arithmetic. Thus $2+3=5$, $2\times3=6$, and $x^2+5x+6\equiv(x+2)\ (x+3)$ are identities modulo m for any positive integer m. Second, we note that the trial-and-error method for finding factors as used in ordinary algebra may be quite difficult because of the large number of possibilities, whereas for a small modulus m the number of possibilities is quite limited.

For example, suppose we wish to know whether $27x^2+159x-21557$ can be factored over the rational number, i.e., whether it can be factored into polynomials with rational coefficients. In A_2 this polynomial is equal to x^2+x+1 (all odd numbers are equal to 1 modulo 2, while all even numbers are equal to 0 modulo 2), and if x^2+x+1 factors, it factors into linear factors. But the only linear polynomials with coefficients in A_2 are x and $x+1$. However, in A_2,

$$x\cdot x\equiv x^2\neq x^2+1,$$

$$x(x+1)\equiv x^2+x\neq x^2+x+1,$$

and

$$(x+1)(x+1)\equiv x^2+2x+1\equiv x^2+1$$
$$\neq x^2+x+1.$$

Hence, $27x^2+159x-21557$ cannot be factored over the rational numbers since it cannot be factored in A_2.

[1] Editor's note: This is the first article of a two-article series on this subject. The second article will appear in THE MATHEMATICS TEACHER for October, 1960.

[2] See, for example, B. W. Jones, "Miniature Number Systems," THE MATHEMATICS TEACHER, LI (1958), 222–231, or W. W. Sawyer, *A Concrete Approach to Abstract Algebra* (San Francisco: W. H. Freeman and Co., 1959), Chapter 3.

It may seem as if the same argument would apply to, for example, $f(x)=24x^3+27x^2+159x-21557$ since this polynomial is also equal to x^2+x+1 in A_2. But this overlooks the possibility that, over the rationals, $f(x)$ might factor into $(ax+b)(cx^2+dx+e)$ where a is an even number and b odd so that, in A_2, $ax+b=1$. Thus, in general, this technique of passing from $f(x)$ over the rational numbers to $f(x)$ over A_m cannot be used if the degree of $f(x)$ over A_m is less than the degree of $f(x)$ over the rationals. (However, we could conclude that $f(x)$ does not have *two* linear factors.)

As a more complicated example, let us ask whether

$$27x^5+40x^4-16x^3-19x^2+100x+45$$

can be factored over the rational numbers. In A_2 this polynomial is equal to x^5+x^2+1. Now obviously, if x^5+x^2+1 factors, one of the factors must be linear or quadratic, since a cubic factor will produce another factor that is quadratic, and a quartic factor will produce another factor that is linear. As before, we observe that the only possible linear factors in A_2 are x and $x+1$. But

$$\begin{aligned} x^5+x^2+1 &\equiv x(x^4+x)+1 \\ &\equiv (x+1)(x^4+x^3+x^2)+1 \end{aligned}$$

in A_2 so that x^5+x^2+1 is not divisible by x or $x+1$ in A_2. Furthermore, the only possible quadratic factors in A_2 are x^2, $x^2+x\equiv x(x+1)$, $x^2+1\equiv(x+1)\ (x+1)$, and x^2+x+1. We do not need to consider the first three possibilities, however, since if, for example, x^5+x^2+1 is divisible by x^2 in A_2, it is certainly divisible by x in A_2—contrary to what we have just shown. Finally, since

$$x^5+x^2+1\equiv(x^2+x+1)(x^3+x^2)+1$$

in A_2, x^5+x^2+1 is not divisible by x^2+x+1 in A_2. (In fact, if we observe that $x^5+x^2+1\equiv x^2(x+1)(x^2+x+1)+1$ in A_2, we may eliminate all possible factors immediately.) Thus x^5+x^2+1 cannot be factored over A_2, and hence

$$27x^5+40x^4-16x^3-19x^2+100x+45$$

cannot be factored over the rational numbers.

While irreducibility (nonfactorability) over A_m implies irreducibility over the rational numbers, the converse is not true. Thus x^2+1 is irreducible over the rational numbers, but reducible (factorable) over A_2, since $x^2+1\equiv(x+1)(x+1)$ in A_2. Similarly,

$$\begin{aligned} f(x) &\equiv 25x^4+15x^3+21x^2-65 \\ &\equiv x^4+x^3+x^2+1\equiv(x+1)(x^3+x+1) \end{aligned}$$

in A_2. Hence, $f(x)$ is reducible over A_2, but we still do not know if it is reducible or irreducible over the rational numbers. However, we may conclude that if $f(x)$ is reducible over the rational numbers, $f(x)$ has a *linear* factor. For if $f(x)\equiv g(x)h(x)$ over the rational numbers where $h(x)$ and $g(x)$ are both quadratic, $f(x)$ will also factor into two quadratic factors over A_2. But since $f(x)\equiv(x+1)(x^3+x+1)$ in A_2, $f(x)$ can factor into quadratic factors in A_2 if and only if x^3+x+1 is factorable in A_2. But

$$x^3+x+1\equiv x(x^2+1)+1\equiv x(x+1)(x+1)+1$$

in A_2 so that x^3+x+1 is irreducible in A_2 since it has a remainder of 1 when divided by x or $x+1$ (the only two linear polynomials in A_2). Having established that if $f(x)$ has a factor, it must have a linear factor, we go to A_3 and observe that, in A_3, the only possible linear factors are x, $x+1$, and $x+2(\equiv x-1)$. But $f(x)\equiv x^4-2\equiv x^4+1$ in A_3, and hence

$$\begin{aligned} f(x) &\equiv x\cdot x^3+1\equiv(x+1)(x^3-x^2+x-1)+2 \\ &\equiv(x-1)(x^3+x^2+x+1)+2 \end{aligned}$$

in A_3. Thus $f(x)$ is not divisible by x, $x+1$, or $x-1$ and is irreducible over A_3. Hence $f(x)$ is irreducible over the rational numbers.

The last example serves to bring out another point. Our analysis in A_3 showed that if $f(x)$ had any factors at all, it had to have a linear factor. Hence even if our results in A_3 had not been conclusive, we would still have known that if we tried all possible linear factors of $f(x)$ over the rationals (namely, $25x \pm 1$, $25x \pm 13$, $5x \pm 1$, $5x \pm 13$, $x \pm 65$, $x \pm 5$, $x \pm 13$, $x \pm 1$) and none of them was actually a factor, there would be no point in looking for a factorization of $f(x)$ into irreducible quadratic factors over the rational numbers. On the other hand, a test over A_2 easily shows that $g(x) \equiv x^4 - 10x^3 + 43x^2 - 90x + 81$ has no linear factors over A_2, and hence none over the rational numbers. Thus our search for factors of $g(x)$ over the rational numbers may be confined to a search for quadratic factors, and we note that, actually, $g(x) = (x^2 - 5x + 9)^2$.

(In checking for the existence of linear factors in A_n one may use to advantage the factor theorem. That is, to find out if $x - a$ is a factor of $f(x)$, we may simply test to see if a is a root of the equation $f(x) = 0$. For example, our testing for the linear factors of $x^4 + 1$ in A_3 could have been accomplished by noting that

$$0^4 + 1 = 1 \neq 0,$$

$$1^4 + 1 = 2 \neq 0,$$

$$(-1)^4 + 1 = 2 \neq 0 \text{ in } A_3.)$$

The *Eisenstein irreducibility criterion* is closely related to the ideas presented above. In stating and in using this theorem, it is convenient to use the symbolism $a \mid b$ (a and b integers) to mean that a divides b, and $a \nmid b$ to mean that a does not divide b.

Let $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$ be a polynomial with integral coefficients. Then $f(x)$ is irreducible over the rational numbers if there exists a prime number p such that $p \nmid a_0$; $p \mid a_1$, $p \mid a_2$, $\cdots$, $p \mid a_n$; and $p^2 \nmid a_n$.

Example 1.

$$16x^6 + 3x^5 + 15x^3 - 12x^2 + 21x + 48$$

is irreducible over the rational numbers. For if we take $p = 3$, we have $3 \nmid 16$; $3 \mid 3$, $3 \mid 0$, $3 \mid 15$, $3 \mid -12$, $3 \mid 21$, $3 \mid 48$; and $3^2 \nmid 48$.

Example 2. $x^n - p$ is irreducible over the rational numbers for any prime p. For $a_0 = 1$, $a_1 = \cdots = a_{n-1} = 0$, $a_n = p$ and thus $p \nmid a_0$; $p \mid a_1, \cdots, p \mid a_n$; and $p^2 \nmid a_n$. (Note that this result shows that $\sqrt[n]{p}[n > 1,\ p \text{ a prime}]$ is always an irrational number.)

Example 3. $x^2 + 1$ is irreducible over the rational numbers. This, of course, is apparent without any need for the Eisenstein theorem, and, as a matter of fact, the Eisenstein theorem cannot be applied directly since we must have $p \mid 1$. One may, however, argue as follows: If $f(x) = x^2 + 1$ were reducible, then so would be

$$f(x+1) \equiv (x+1)^2 + 1 \equiv x^2 + 2x + 2.$$

Now for $f(x+1)$ we take $p = 2$ and have $2 \nmid 1$, $2 \mid 2$, and $2^2 \nmid 2$. Hence $f(x+1)$ is irreducible over the rational numbers, and therefore $f(x)$ is likewise irreducible over the rational numbers.

A proof of the Eisenstein theorem and additional discussion of these topics may be found in B. L. van der Waerden's *Modern Algebra*.[3]

[3] Vol. 1 (New York: Ungar Publishing Co., 1953).

Minimum multiplying magic squares

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Here are magic squares constructed on products rather than on sums.

MAGIC SQUARES of the arithmetic type, shown in Figure 1, have been a source of fascination since ancient times. They have engaged the attention of the mystic, the mathematician, and the amateur dilettante alike. An arithmetic magic square is a numerical square array composed of different positive integers, in which every row, column, and diagonal add to the same number. In this article, we shall discuss magic squares of the *geometric* or *multiplying* type in which each row, column, and diagonal multiply out to the same number. We shall show how they may be generated directly from the simple arithmetic magic square by rules of elementary algebra. Then we will develop a family of minimum geometric magic squares in which each row, column, and diagonal multiplies out to the smallest possible integer consistent with the order of the square. General construction rules and formulae are offered for these squares.

DEVELOPMENT OF THE ELEMENTARY MAGIC SQUARE—ORDER 3

Let us derive the elementary magic square of Figure 1, since the arithmetic models are required to develop our geometric squares. We formulate the problem: Arrange the integers from 1 to 9 in a magic square such that each row, column, and diagonal add to the same number.

Figure 1

Arithmetic Magic Square

8	1	6
3	5	7
4	9	2

Solution: Let n = the *order* of the magic square, and let n^2 be the number of terms in the square The arithmetic sum of these numbers must be:

$$S=\frac{n^2}{2}(n^2+1).$$

For $n=3$, $S=\frac{9}{2}(10)=45$

Each row, column, and diagonal must add to N where:

$$N=\frac{S}{n}=\frac{n}{2}(n^2+1)$$

For $n=3$, $N=15$

a	b	c
d	e	f
g	h	i

Figure 2

Assume the unknowns are as shown in Figure 2. Add the middle row, middle column, and both diagonals as follows:

$$(d+e+f)+(b+e+h)+(a+e+i)+(c+e+g)=4(15)$$

$$(a+b+c+d+e+f+g+h+i)+3e=60$$

Therefore,

$$45+3e=60$$

$$e=5.$$

Therefore, the middle number must be 5. In fact, if any other integer is placed in the center, the sum of 15 cannot be obtained.

Now let us consider where to place the number 1. There are really only two possible configurations. One configuration results by placing 1 in the middle slot for either a row or column, and the second configuration from placing 1 in any corner of the square. If we rotate the square, we can see that any middle slot will give rise to the same configuration as any other middle slot. The same is true if 1 is placed in any corner.

Let us select the incorrect solution by placing 1 in a corner. Immediately 9 falls into the opposite corner, as in Figure 3. Now consider the pair c and f, which must total 6. Since both c and f cannot each be the number 3, the only possibilities are 2 and 4. However, c can be neither 2 or 4 since b must then be either 12 or 10. Thus 1 cannot be placed in a corner. Therefore 1 must be in a middle slot. Let $b=1$ and $h=9$.

1	b	c
d	5	f
g	h	9

Figure 3

a	1	c
d	5	f
4	9	2

Figure 4

Now consider numbers g and i. Once more the only possibilities are 2 and 4, and it is true that either $g=2$, $i=4$ or $g=4$ and $i=2$ gives a proper solution. Assume, as in Figure 4, that $g=4$ and $i=2$. Then the remaining numbers are easily found as $a=8$, $c=6$, $d=3$, and $f=7$, which results in the array of Figure 1.

It is clear that a valid solution would have resulted if 2 and 4 were interchanged. The new magic square would simply be the mirror image of the one shown.

Arithmetic square variation

An interesting and useful variation of this square is obtained by staggering the numbers in sets of three as shown in Figure 5c. A new magic square is obtained by superimposing the original magic square with a correction magic square shown in Figure 5. Staggered squares may also be constructed for $n=4$ and $n=5$ as shown in Figures 6 and 7.

Higher order arithmetic squares

The arithmetic and correction squares shown in Figures 5, 6 and 7 will be used with geometric squares developed later. The reader will find a definite pattern in each case if he follows the numbers consecutively, starting with the number 1. In fact, the odd order squares 7, 9, 11, etc., can be written down by inspection once the pattern for the magic square of order 5 is mastered. The same is true of the correction squares in each case. An exhaustive treatise on this subject is given by W. S. Andrews in Reference 1.

Geometric magic squares

Suppose one is now given this problem: Using the elementary additive magic square of Figure 1, generate a geometric, or multiplying magic square, choosing any nine different integers to fill the spaces.

In these days of binary systems and digital computers an ingenious thought comes to mind. Suppose each number in Figure 1 is regarded as the exponent of a common base B. Then the sum of these exponents, 15 in this case, must be the same for each row, column, and diagonal. Furthermore the *common product*, P, of the numbers in each row, column, and diagonal must be the same, namely B^{15}. Using the arithmetic square of Figure 1 for the exponents and letting the base $B=2$, the resulting multiplication square is shown in Figure 8. The common product P is obviously 2^{15} or 32,768. To reduce the size of the numbers, divide each term by 2.

The geometric magic square of Figure 9 results. This result could, however, have been derived more quickly by using numbers from 0 to 8 in the original magic square.

STAGGERED TECHNIQUE (3×3)

Original (15)			+	Correction (3)			=	Result (18)		
8	1	6		2	0	1		10	1	7
3	5	7		0	1	2		3	6	9
4	9	2		1	2	0		5	11	2

Figure 5a *Figure* 5b *Figure* 5c

STAGGERED TECHNIQUE (4×4)

Original (34)				+	Correction (6)				=	Result (40)			
1	14	15	4		0	3	3	0		1	17	18	4
8	11	10	5		1	2	2	1		9	13	12	6
12	7	6	9		2	1	1	2		14	8	7	11
13	2	3	16		3	0	0	3		16	2	3	19

Figure 6a *Figure* 6b *Figure* 6c

Arithmetic Square, 5×5

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Sum = 65

Figure 7a

Correction Square, 5×5

3	4	0	1	2
4	0	1	2	3
0	1	2	3	4
1	2	3	4	0
2	3	4	0	1

Sum = 10

Figure 7b

Geometric Magic Square

2^8	2^1	2^6
2^3	2^5	2^7
2^4	2^9	2^2

Figure 8

Binary Numbers			Base Ten Numbers		
2^7	2^0	2^5	128	1	32
2^2	2^4	2^6	4	16	64
2^3	2^8	2^1	8	256	2

$P = 2^{12} = 4096$

Figure 9

Arithmetic Correction

2	0	1
0	1	2
1	2	0

Sum = 3

Figure 10

Geometric Correction

$(3/8)^2$	$(3/8)^0$	$(3/8)^1$
$(3/8)^0$	$(3/8)^1$	$(3/8)^2$
$(3/8)^1$	$(3/8)^2$	$(3/8)^0$

$P = (3/8)^3$

Figure 11

Original			×	Correction			=	Minimum		
128	1	32		9/64	1	3/8		18	1	12
4	16	64		1	3/8	9/64		4	6	9
8	256	2		3/8	9/64	1		3	36	2

$P_1 = 2^{12}$ $P_2 = (3/8)^3$ $P = P_1P_2 = 6^3 = 216$

Figure 12a *Figure* 12b *Figure* 12c

Minimum, 4×4

1	54	108	8
24	36	18	3
72	12	6	9
27	2	4	216

$P = 6^4 = 46{,}656$

Figure 13a

Binary, 4×4

1	2^{13}	2^{14}	2^3
2^7	2^{10}	2^9	2^4
2^{11}	2^6	2^5	2^8
2^{12}	2^1	2^2	2^{15}

$P = 2^{30} \approx 10^9$

Figure 13b

Minimum geometric magic square

One now queries whether 4,096 is the smallest possible product using different positive integers. The answer is no.

Suppose one considered the staggered addition square of Figure 10 to be the model for a correction multiplication square with the common base 3/8, as shown in Figure 11.

The common product of this correction geometric magic square is $(3/8)^3$. Now apply this correction square to the geometric magic square of Figure 9, and there results a new *minimum* magic square, as shown in Figure 12c. Clearly, the factor 3/8 is chosen so that the numbers 8, 16, 32, 64, 128, and 256 of the original square are replaced by 3, 6, 12, etc.

The last geometric magic square in Figure 12 has a common product of only 216 and uses six of the integers from 1 to 9! Clearly the prime numbers 5 and 7 cannot be used since they cannot be factored into smaller integers. Without details of rigorous proof, we submit that the minimum array of Figure 12c is indeed the smallest possible magic multiplication square. We will now show that the correction principle of construction shown in Figures 10, 11, and 12 apply to any order square.

Minimum, 5×5

54	648	1	12	144
324	16	6	72	27
8	3	36	432	162
48	18	216	81	4
9	108	1296	2	24

$P = 6^{10} \approx 6 \times 10^7$

Figure 14a

Binary, 5×5

2^{16}	2^{23}	1	2^7	2^{14}
2^{22}	2^4	2^8	2^{13}	2^{15}
2^3	2^5	2^{12}	2^{19}	2^{21}
2^9	2^{11}	2^{18}	2^{20}	2^2
2^{10}	2^{17}	2^{24}	2^1	2^8

$P = 2^{60} \approx 10^{18}$

Figure 14b

Minimum 3×3

2^1	2^0	2^2
2^2	2^1	2^0
2^0	2^2	2^1

$P_1 = 2^3$

Figure 15a

3^2	3^0	3^1
3^0	3^1	3^2
3^1	3^2	3^0

$P_2 = 3^3$

Figure 15b

$3^2 \cdot 2^1$	$3^0 \cdot 2^0$	$3^1 \cdot 2^2$
$3^0 \cdot 2^2$	$3^1 \cdot 2^1$	$3^2 \cdot 2^0$
$3^1 \cdot 2^0$	$3^2 \cdot 2^2$	$3^0 \cdot 2^1$

$P = P_1 P_2 = 2^3 \cdot 3^3 = 6^3$

Figure 15c

Multiplication squares of higher order

Using as models the arithmetic and the correction squares shown in Figures 6 and 7, and applying the geometric principles developed in the previous section, one quickly arrives at the minimum magic squares of Figure 13a and Figure 14a for $n=4$ and $n=5$. The straight binary types are shown in Figure 13b and 14b for comparison.

Each of the minimum squares uses *seven* of the first nine integers, the prime numbers 5 and 7 being omitted in each case. Although the minimum geometric squares of Figures 12c and 14a are shown in Reference 2, the author did not show the minimum squares of Figures 13a and 18, nor did he offer any rules or formulae for the construction of minimum multiplying magic squares. We shall now show that a minimum multiplying square of any order n may be constructed by a simple procedure involving the product of two elementary multiplying squares for each case.

Rules for construction for minimum squares of order n

1) Construct an elementary multiplying square in powers of 2, with each exponent placed in the order shown in Figures 15a, 16a, and 17a. These exponents are put down in the same order as the consecutive numbers in the corresponding arithmetic squares of Figures 5a, 6a, and 7a. Typically, in the 5×5 square, the consecutive exponents read: 0, 1, 2, 3, 4; 0, 1, 2, 3, 4, etc.

2) Then construct a similar multiplying square in powers of 3, with each exponent placed in the order shown in Figures 15b, 16b, and 17b. These exponents are put down in the same order as the corresponding correction squares shown in Figures 5b, 6b, and 7b. In the 5×5 square, the consecutive exponents read:

MINIMUM 4×4

2^0	2^1	2^2	2^3
2^3	2^2	2^1	2^0
2^3	2^2	2^1	2^0
2^0	2^1	2^2	2^3

$P_1 = 2^6$

Figure 16a

3^0	3^3	3^3	3^0
3^1	3^2	3^2	3^1
3^2	3^1	3^1	3^2
3^3	3^0	3^0	3^3

$P_2 = 3^6$

Figure 16b

$2^0 \cdot 3^0$	$2^1 \cdot 3^3$	$2^2 \cdot 3^3$	$2^3 \cdot 3^0$
$2^3 \cdot 3^1$	$2^2 \cdot 3^2$	$2^1 \cdot 3^2$	$2^0 \cdot 3^1$
$2^3 \cdot 3^2$	$2^2 \cdot 3^1$	$2^1 \cdot 3^1$	$2^0 \cdot 3^2$
$2^0 \cdot 3^3$	$2^1 \cdot 3^0$	$2^2 \cdot 3^0$	$2^3 \cdot 3^3$

$P = P_1 P_2 = 2^6 \cdot 3^6 = 6^6$

Figure 16c

MINIMUM, 5×5

2^1	2^3	2^0	2^2	2^4
2^2	2^4	2^1	2^3	2^0
2^3	2^0	2^2	2^4	2^1
2^4	2^1	2^3	2^0	2^2
2^0	2^2	2^4	2^1	2^3

$P_2 = 2^{10}$

Figure 17a

3^3	3^4	3^0	3^1	3^2
3^4	3^0	3^1	3^2	3^3
3^0	3^1	3^2	3^3	3^4
3^1	3^2	3^3	3^4	3^0
3^2	3^3	3^4	3^0	3^1

$P_1 = 3^{10}$

Figure 17b

$3^3 \cdot 2^1$	$3^4 \cdot 2^3$	$3^0 \cdot 2^0$	$3^1 \cdot 2^2$	$3^2 \cdot 2^4$
$3^4 \cdot 2^2$	$3^0 \cdot 2^4$	$3^1 \cdot 2^1$	$3^2 \cdot 2^3$	$3^3 \cdot 2^0$
$3^0 \cdot 2^3$	$3^1 \cdot 2^0$	$3^2 \cdot 2^2$	$3^3 \cdot 2^4$	$3^4 \cdot 2^1$
$3^1 \cdot 2^4$	$3^2 \cdot 2^1$	$3^3 \cdot 2^3$	$3^4 \cdot 2^0$	$3^0 \cdot 2^2$
$3^2 \cdot 2^0$	$3^3 \cdot 2^2$	$3^4 \cdot 2^4$	$3^0 \cdot 2^1$	$3^1 \cdot 2^3$

$P = P_1 P_2 = 3^{10} \cdot 2^{10} = 6^{10}$

Figure 17c

MINIMUM 6×6 MULTIPLYING SQUARE

$3^0 \cdot 2^0$	$3^5 \cdot 2^4$	$3^5 \cdot 2^3$	$3^0 \cdot 2^2$	$3^5 \cdot 2^1$	$3^0 \cdot 2^5$
$3^4 \cdot 2^5$	$3^1 \cdot 2^1$	$3^4 \cdot 2^3$	$3^4 \cdot 2^2$	$3^1 \cdot 2^4$	$3^1 \cdot 2^0$
$3^3 \cdot 2^5$	$3^3 \cdot 2^4$	$3^3 \cdot 2^2$	$3^3 \cdot 2^3$	$3^3 \cdot 2^1$	$3^3 \cdot 2^0$
$3^2 \cdot 2^0$	$3^2 \cdot 2^4$	$3^3 \cdot 2^2$	$3^3 \cdot 2^3$	$3^3 \cdot 2^1$	$3^2 \cdot 2^5$
$3^1 \cdot 2^5$	$3^4 \cdot 2^1$	$3^1 \cdot 2^2$	$3^1 \cdot 2^3$	$3^4 \cdot 2^4$	$3^4 \cdot 2^0$
$3^5 \cdot 2^0$	$3^0 \cdot 2^1$	$3^0 \cdot 2^3$	$3^5 \cdot 2^2$	$3^0 \cdot 2^4$	$3^5 \cdot 2^5$

$P = 3^{15} \cdot 2^{15} = 6^{15}$

Figure 18

0, 0, 0, 0, 0; 1, 1, 1, 1, 1; 2, 2, 2, 2, 2; etc.

3) Finally, multiply the first elementary square by the second elementary square, term for term. This may be done by simply writing down both squares side by side, as shown in Figures 15c, 16c, and 17c. It will be noted from Figures 15c, 16c, and 17c that the resulting common product, P, of the minimum square is equal to:

$$P = 3^{n(n-1)/2} \cdot 2^{n(n-1)/2} = 6^{n(n-1)/2}$$

A minimum 6×6 multiplying square is given in Figure 18 and was written directly by applying the third step only.

In fact, any minimum square can be written down directly by combining the three steps into one operation. The reader is invited to verify Figure 18 and to construct a 7×7 minimum square by the rules given. The common product is $P = 6^{21}$.

General formulae

It can be seen that a family of minimum multiplying squares can be generated by the rules given for any n. The common product, P_m, of the minimum square is given by:

$$P_m = 6^{n(n-1)/2}$$

For comparison, the common product, P_b, for the elementary binary squares (illustrated in Figures 13b and 14b) is:

$$P_b = 2^{n(n^2-1)/2}$$

The reduction factor, F, resulting from these techniques is:

$$F = \frac{Pb}{Pm} = \left[\frac{(2^n)}{3}\right]^{n(n-1)/2}$$

The reduction factor, F, increases rapidly with n, varying from only about 19 for $n = 3$ to about 19 billion for $n = 5$ (see Figure 14).

Conclusions

The subject of minimum multiplying squares is relatively new. It is believed that the principles of construction and the formulae presented here are entirely original and have never appeared in print before, nor has the family of minimum squares presented in Figures 15 through 18.

It is shown that a family of minimum multiplying magic squares of order n may be constructed by using suitable staggered multiplying squares based on the corresponding arithmetic models. In fact, minimum squares of any odd order may be written directly by inspection.

References

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2. H. E. Dudeney, *Amusements in Mathematics* (New York: Dover Publications, 1958).
3. W. W. R. Ball, *Mathematical Recreations and Essays*, 11th ed. (New York: The Macmillan Company, 1939).

A yearbook on teaching the gifted

The National Council of Teachers of Mathematics is presently planning a yearbook on materials and methods for teaching the gifted. Any suggestions concerning appropriate contributors and contributions for this yearbook would greatly be appreciated by the committee in charge. Please send suggestions to Julius H. Hlavaty, Director of Mathematics Commission Program, Commission on Mathematics, 425 West 117th Street, New York 27, New York.

Mathematical induction, Σi^p and factorial powers

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The algebra student asks, "If we know the formula for the sum of the integers, why do we have to prove it?"

MATHEMATICAL INDUCTION, highly recommended by all mathematicians, is currently one of the more annoying and hence avoided topics in algebra. By its nature it always appears as a special topic outside the orbit of the usual plug-in techniques of the rest of the algebra course. The method of proof seems contrived, long, unnatural, and often confusing. This paper is an attempt to discuss mathematical induction in a natural, obvious setting which will make the topic clearer and hence more useful.

We recall that in the usual approach the principle of mathematical induction is announced and then used to prove various formulas of the sort

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

The student is mystified. The formula is given; why bother to prove it? Going through the manipulations of the induction proof does not do much more to convince the student of the truthfulness of the formula than does his observation that the formula is true for several values. The real question which fascinates the student is, "How does the author know what the polynomial formula is to be proved?" This, the intriguing part of the topic, is left as much a mystery at the end of the chapter as it was at the beginning.

The heart of this proposed approach is to capture this interest and to show that the interesting question can be solved by simple plug-in techniques. Further, this leads naturally to mathematical induction as an obvious, necessary step to convince the student of the correctness of the newly discovered formula. We show the method by considering a few specific examples.

Consider the question, "Is it possible to express $\sum_{i=0}^{n} i$ as a polynomial in n of degree less than or equal to 4?" We recognize that if we cannot find a polynomial of degree 4, we might still be able to find a polynomial of degree 5 or degree 50. More specifically we ask, "Can we find coefficients a_0, a_1, a_2, a_3, a_4 such that

$$(1)\quad \sum_{i=0}^{n} i = a_0 + a_1 n^{(1)} + a_2 n^{(2)} + a_3 n^{(3)} + a_4 n^{(4)}$$

is a true statement for all nonnegative integers n?"

The polynomial in the right member of (1) uses "factorial powers" of n rather than ordinary powers. The factorial powers are defined

$$n^{(1)} = n,\ n^{(2)} = n(n-1),\ \cdots,\ n^{(l)} = n(n-1)(n-2)\cdots(n-l+1).$$

That the right member of (1) is a polynomial of degree no greater than 4 is easily seen by multiplying out the factorial powers and collecting terms. Factorial powers are used in place of ordinary powers to simplify the algebra. The simplification

TABLE 1

n	$\sum_{i=0}^{n} i$	$n^{(1)}=n$	$n^{(2)}=n(n-1)$	$n^{(3)}$	$n^{(4)}=n(n-1)(n-2)(n-3)$
0	0	0	0	0	0
1	1	1	0	0	0
2	3	2	2	0	0
3	6	3	6	6	0
4	10	4	12	24	24
5	15	5	20	60	120

comes because, for example, $x^{(4)}=0$ for $x=0, 1, 2, 3$, as can be seen by direct substitution.

We first get some necessary conditions on the coefficients a_j. If (1) is to be true for all n, then (1) must be true in particular for $n=0$, for $n=1$, for $n=2$, $\cdots$. We make a convenient table of values and substitute into (1) as shown in Table 1.

Substituting into (1) yields the following equations which must be satisfied by $a_0, \cdots, a_4$.

$$0=a_0$$
$$1=a_0+a_1$$
$$3=a_0+2a_1+2a_2$$
$$6=a_0+3a_1+6a_2+6a_3$$
$$10=a_0+4a_1+12a_2+24a_3+24a_4.$$

These simultaneous equations are easily solved successively. We get $a_0=0$, $a_1=1$, $a_2=1/2$, $a_3=0$, and $a_4=0$.

Hence, if there is a polynomial of degree less than or equal to 4 equal to $\sum_{i=0}^{n} i$, it can be none other than

$$\sum_{0}^{n} i=n^{(1)}+\tfrac{1}{2}n^{(2)}, \tag{2}$$

since this is the only such polynomial satisfied by $n=0, 1, 2, 3, 4$.

It is interesting that few people will believe that (2) is a true statement for all n when they approach it from this point of view. Instinctively one feels that it is just luck that a_3 and a_4 turned out to be zero. Intuition says that surely (2) will not be true for all n, or that some more terms of higher degree will have to be added if the formula is to be valid. It is to settle this uneasiness that mathematical induction is introduced.

The principle of mathematical induction is as follows: if a set of positive integers contains the integer 1, and contains the integer $k+1$ if it contains the integer k, then the set contains all the positive integers. This principle is considered a perfectly obvious property of the integers and is usually taken as one of the axioms describing the integers.

We ask now for the set of integers, n, which satisfy (2). We have shown that 0, 1, 2, 3, 4 are in the set. We seek to show that $(k+1)$ is in the set if k is. That is, that

$$\sum_{i=0}^{i=k+1} i=(k+1)+\tfrac{1}{2}(k+1)k \tag{3}$$

is true if

$$\sum_{i=0}^{i=k} i=k+\tfrac{1}{2}k(k-1). \tag{4}$$

Subtracting (4) from (3), we get the necessary condition

$$k+1=1+\tfrac{1}{2}k[(k+1)-(k-1)]. \tag{5}$$

But (5) is clearly true. Further, we can go backward, adding the corresponding sides of (5) and (4) to get (3). Hence if (2) is true for $n=k$, (2) is true for $n=k+1$. Hence by the principle of mathematical induction, (2) is true for all positive integers n.

Expanding $n^{(2)}$ in (2) gives as an alternative form

$$\sum i = \frac{n(n+1)}{2} \cdot \tag{6}$$

We consider another special case. Can we find coefficients $b_0, b_1, \cdots b_s$, such that

$$\sum_{i=0}^{n} i^3 = 0^3+1^3+2^3+\cdots+n^3 = b_0 + b_1 n^{(1)} + b_2 n^{(2)} + \cdots + b_s n^{(s)} \tag{7}$$

is true for all nonnegative integers n, and where s, the degree of the polynomial, must also be determined?

Proceeding as before, we find the coefficients b_j must satisfy the following equations:

$$\begin{aligned}
0 &= b_0 \\
1 &= b_0 + b_1 \\
9 &= b_0 + 2b_1 + 2b_2 \\
36 &= b_0 + 3b_1 + 6b_2 + 6b_3 \\
100 &= b_0 + 4b_1 + 12b_2 + 24b_3 + 24b_4 \\
225 &= b_0 + 5b_1 + 20b_2 + 60b_3 + 120b_4 + 120b_5 \\
441 &= b_0 + 6b_1 + 30b_2 + 120b_3 + 360b_4 + 720b_5 + 720b_6.
\end{aligned} \tag{8}$$

Solving successively, we find that $b_0=0$, $b_1=1$, $b_2=\frac{7}{2}$, $b_3=2$, $b_4=\frac{1}{4}$, $b_5=0$, $b_6=0$.

The advantage of the factorial power notation is particularly obvious here. Equations in (8) would be an indefinite number of equations in an indefinite number of unknowns if ordinary powers were used. This would be an impossible situation for the elementary algebra student.

Hence if there is a polynomial equal to $\sum_{i=0}^{n} i^3$ for all n, it is interesting to conjecture that it might be

$$\sum^{n} i^3 = n^{(1)} + \tfrac{7}{2} n^{(2)} + 2n^{(3)} + \tfrac{1}{4} n^{(4)}, \tag{9}$$

since b_5 vanished. It is, of course, quite possible that more terms will be necessary. We seek the set of integers, n, which makes (9) true. By trial we have seen that 0, 1, 2, 3, 4, 5 are in the set. We ask, "Is $k+1$ in the set if k is?" that is, is

$$\sum^{k+1} i^3 = (k+1)^{(1)} + \tfrac{7}{2}(k+1)^{(2)} + 2(k+1)^{(3)} + \tfrac{1}{4}(k+1)^{(4)} \tag{10}$$

if

$$\sum^{k} i^3 = k^{(1)} + \tfrac{7}{2} k^{(2)} + 2k^{(3)} + \tfrac{1}{4} k^{(4)}? \tag{11}$$

Subtracting corresponding sides, we get the necessary condition

$$\begin{aligned}
(k+1)^3 &= 1 + \tfrac{7}{2} \cdot 2k^{(1)} + 2 \cdot 3k^{(2)} + \tfrac{1}{4} \cdot 4k^{(3)} \\
&= 1 + 7k + 6k(k-1) + k(k-1)(k-2),
\end{aligned} \tag{12}$$

which is easily seen to be an identity by multiplying out the two sides. Adding the corresponding sides of (12) and (11) gives (10). Hence by the principle of mathematical induction, (9) is true for all positive n.

In making subtractions like the one above, the simple identity

$$(k+1)^{(h)} - k^{(h)} = hk^{(h-1)} \tag{13}$$

is very useful. Step (9) can be written in ordinary powers instead of factorial powers as

$$\sum i^3 = \left\{ \frac{n(n+1)}{2} \right\}^2 \cdot \tag{14}$$

These two examples suggest that $\sum_{i=0}^{n} i^p$ can be expressed as a polynomial in n of degree $p+1$. This is a true theorem. The polynomial can be derived in any particular case by the above method. This method does not appear to lead to a simple proof of the general theorem. Such proofs may be found in the theory of difference equations, but this is usually too unfamiliar to elementary algebra students.

The lower limit in $\sum i^p$ is taken as 0 rather than 1 to make it easy to show that a_0 (or b_0) $=0$. The value of $\sum i^p$ is the same in both cases.

Training of mathematics teachers in the USSR

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This is a description of the education of today's mathematics teachers in the Soviet Union.

SOVIET EDUCATIONAL LEADERS believe that more than any other factor the key element to effective instruction is the well-trained teacher. Toward this end they have developed a thorough program of preservice and in-service education for teachers at all grade levels.

As a member of an official three-man U. S. Office of Education delegation which studied the schools and the teacher-training institutions in the Soviet Union for twenty-eight days in May and June of 1959, the author visited seven teachers' colleges located in Moscow, Kiev, Tbilisi, and Leningrad.[1] This article is a brief report of observations made concerning the training of mathematics teachers.

RESPONSIBILITY FOR AND CONTROL OF TEACHER EDUCATION

Primary responsibility for teacher education rests in the respective ministries of the fifteen republics. The Ministry of Higher and Secondary Special Education of the USSR has a supervisory responsibility in that it must approve the curriculum and insure that the program of instruction is of high quality. Annual admission quotas for the regular day programs are established for the entire USSR and for each institution separately. Not one student in excess of the quota is officially admitted to any institution. There are no fixed quotas, however, for evening or correspondence study programs, which enroll forty-three per cent of all the students in higher education.

Since there are not enough places in the regular day programs for all those who would like to become teachers, applicants must compete with one another for admission. This situation is true for all types of higher education in the Soviet Union. A student is permitted to apply to only one institution in any one year, and he must undergo a series of entrance examinations which include Russian language and literature, the foreign language studied in the secondary school, and the specialties in which the student plans to major. The applicants who score highest in the examinations are admitted up to the limits of the quota, and those who are unsuccessful may reapply another year at the same or another institution.

Uniformity of programs of instruction throughout the Soviet Union is achieved through the curriculums, or "study plans" (uchebnye plany), that are promulgated by the ministries of education. Sometimes in actual practice, however, local training facilities and needs may require slight modifications of official plans.

TYPES OF TEACHER TRAINING INSTITUTIONS

In imperial Russia there were few teacher-training colleges. Today, how-

[1] The other two members of the Office of Education delegation were Dr. William K. Medlin, Specialist in Comparative Education for Eastern Europe, Division of International Education, and Dr. Marshall L. Schmitt, Specialist in Industrial Arts, Division of State and Local School Systems. A detailed report, *Soviet Educational Programs*, is being published and will be available late this year from the Superintendent of Documents, U.S. Government Printing Office, Washington 25, D.C.

ever, teacher-training institutions constitute one of the largest groups of colleges.

Teachers for kindergarten and grades I–IV are prepared at pedagogical schools of two types. The two-year school (now the main type), of which there are approximately 280 is based on ten years of general education, while the four-year school is based on seven years of general education. Under the new educational reforms being adopted throughout the USSR, elementary teachers in the future will be prepared in a four-year curriculum following ten (eventually eleven) years of primary and secondary education. Since elementary teachers teach all the subjects given at their grade levels, they are given a broad training, including some mathematics, and no specialization.

Secondary teachers, or teachers for grades V through XI, are trained as subject-matter specialists. They are educated at the five-year pedagogical institutes based on ten years of general education.[2] In 1958–59 there were a total of 212 pedagogical institutes with an attendance of 515,000.

In addition to the five-year pedagogical institutes, the universities, particularly since 1955, have supplied increasing numbers of secondary teachers for single subject work. The development in Soviet teacher-training concepts over the past decade has been directed toward the ever-increasing preparation for teachers, especially at the secondary level. Evidently this policy has led to a lengthening and an intensification of the preparatory program and to greater use of university facilities for teacher education.

To increase the competency of teachers already in service and to enable them to keep abreast of latest developments in subject-matter content and pedagogy, in-service institutes for teachers have been established throughout the USSR. Within every five-year period a teacher is expected to attend such an institute, either during the regular academic-year program in which he is given one day off from his teaching to attend, or during the summer program. In 1958–59 a total of 3,200 teachers of the 23,000 teachers in Leningrad attended one of the two in-service institutes in that city. In addition to the regular academic-year and summer courses, seminars and conferences are regularly held in all fields including mathematics.

[2] Prior to 1957, the teacher education program for secondary teachers was of four years duration.

Mathematics curriculums in pedagogical institutes

Since most specialized teachers of mathematics are educated at the pedagogical institutes, the remainder of this article will be devoted to a discussion of mathematics programs given at this type of institution.

Each student enrolled in the regular day program of a pedagogical institute must take a double major in related subjects. If a student wants to become a mathematics teacher, he or she must also take either physics or technical drawing as a comajor. The curriculum required of a student specializing in mathematics and physics is shown in the accompanying outlines.[3]

It is seen upon examination of this curriculum that the student spends more than half of the course work time in the study of advanced topics in mathematics and physics. However, professional education in the art of teaching is not slighted but is an important part of the curriculum. Not only must each student take courses in psychology and in the theory and history of pedagogy, but he or she must also take special methods in the teaching of mathematics and special methods in the teach-

[3] A similar curriculum for students specializing in mathematics and technical drawing will be found in W. K. Medlin and N. Apanasewicz, *Programs in Secondary Teacher Education in the USSR* (U. S. Department of Health, Education, and Welfare, Office of Education, International Educational Relations Publication No. 9 [Washington 25, D. C., December 1958]).

CURRICULUM FOR MAJORS IN MATHEMATICS AND PHYSICS

Subject	Hours			Semesters	Reports	Exams
	Lecture	*Laboratory*	*Practical exercises*			
1. History of the Communist party of the Soviet Union	120	—	104	1, 2, 3, 4	2	2
2. Political economy	98	—	42	5, 6, 7, 8	2	2
3. Dialectical and historical materialism	80	—	60	7, 8, 9	1	2
4. Psychology	50	—	34	1, 2	—	1
5. Pedagogy	70	—	50	3, 4	1	1
6. School hygiene	18	—	18	6	1	—
7. History of pedagogy	54	—	18	5, 6	—	1
8. Mathematical analysis	204	—	204	1, 2, 3, 4	4	4
9. Analytic geometry	86	—	86	1, 2	1	2
10. Projective and descriptive geometry	78	—	32	5, 6	1	1
11. Foundations of geometry	54	—	10	9	—	1
12. Higher algebra	104	—	88	1, 2, 3	2	2
13. Theory of numbers	36	—	12	6	—	1
14. Foundations of arithmetic	36	—	—	7	—	1
15. Theory of functions of a real variable	50	—	—	6	—	1
16. Theory of functions of a complex variable	54	—	—	5	—	1
17. Special seminar in mathematics or special practical training in physics	—	—	84	8, 9	2	—
18. Elementary mathematics (structure and concepts)	180	16	204	1, 2, 3, 4, 5, 6, 7	4	3
19. Methods of teaching mathematics	92	—	102	6, 7, 8, 9	3	1
20. Special practical training (mathematical models, surveying, calculation)	—	92	—	7, 8, 9	3	—
21. Mechanical drawing	18	68	—	1, 2	2	—
22. General physics	272	204	144	2, 3, 4, 5	4	4
23. Theoretical mechanics	56	—	56	7, 8	1	1
24. Astronomy	54	18	—	9	—	1
25. Methods of teaching physics	60	100	50	6, 7, 8, 9	3	1
26. Practical training in school workshops with elements of technology of materials	32	150	6	4, 5, 6	3	—
27. Electrical and radio engineering	48	80	12	7, 8	2	—
28. Machine technology with practical training in auto-tractor technology	54	130	6	8, 9	2	—
29. Educational movies	—	36	—	7	1	—
30. Special course (elective)	54	—	—	9	1	—
31. Foreign language	—	—	140	1, 2, 3, 4	3	1
32. Physical education	—	—	140	1, 2, 3, 4	4	—
33. Special training	—	—	48	5	1	—
34. Pedagogical practice in extracurricular activity	—	—	60	5	1	—
	2114 (*sic*)	894	1810			

Total hours: *4818*

Teaching practice: School—6 weeks in semester 7; 10 weeks in semester 10.
Industrial—2 weeks in semester 10.

ing of physics, as well as sixteen full weeks of supervised teaching in these subjects during the fourth and fifth years in one of the schools in the area near the pedagogical institute or in demonstration schools of the institutes which have them.

Only a small portion of the student's time is devoted to the humanistic and social studies areas, the assumption being that the student has already received fairly adequate education in these subjects in the secondary school. In spite of having been required to study a foreign language for six years in the secondary school, each student is nevertheless compelled to spend two additional years of foreign language study in the same language or a new language. A student may elect to continue the study of foreign language beyond the required two years. Physical education is also obligatory for all students during the first two years.

The chief difference between the curriculums of the regular day program and the evening and correspondence study programs of the pedagogical institutes is that while two fields of specialization are required of day students, only one is required of other students. However, graduation from any of these programs is considered as being of equal merit.

Methods of instruction

In Moscow, the author had an opportunity to visit a second-year mathematical analysis class at the state pedagogical institute named after Lenin that has over 4,000 day students. This class, consisting of about ninety students, almost all of whom were women, was attending a lecture on linear differential equations given by their professor who had a doctorate degree awarded by the University of Moscow.

The professor lectured, and the students took notes. The students were permitted to interrupt and ask questions. As in all courses, the period of instruction lasted for two academic hours of fifty minutes each, separated by a five-minute rest interval. Each student had a textbook for the course. This textbook, which was published by the state publishing house, as all textbooks are, was purchased and owned by the student. At the conclusion of the second class session a reading and exercise assignment was given, to be completed before the next meeting of the class.

From time to time, quizzes are given on the material that has been covered. Students are always marked on the following basis: 5 (excellent), 4 (good), 3 (satisfactory), 2 (unsatisfactory), and 1 (failing). At the end of the semester final course examinations are given in all subjects taken by the students. These examinations are given over a period lasting from three to four weeks. Each student receives a final course grade of 5, 4, 3, 2, or 1, which is then recorded in the student's record book as well as in the office of the institute.

Examples of optional courses for majors in mathematics and physics

Practical training in extracurricular and extra-school activity	100 hr.
Methods of mathematical physics	40
Differential geometry	40
Theory of probability	40
Methods of approximate calculation	40
Algorithms and computing machines	40
Modern algebra	40
Nuclear physics	40
Physics of semiconductors and dielectrics	40
Aerodynamics	40
History of physics	40
History of mathematics	40
Selected topics in elementary mathematics	40
Special topics in the methodology of physics	40
Improving sports skills	420
Foreign language (in years III and IV)	140
Choral singing	250
Individual instruction in playing musical instruments	250

Study load and stipends

The students have scheduled classes for no more than 36 hours a week in the first three years, and 28 to 30 hours a week in the fourth and fifth years. Students come to the institute six days a week for four to

six hours a day during the first two years. In the last three years, they come five days a week for four to six hours a day, thus giving them an extra free day besides Sundays.

Most of the students attend school on stipends that are usually sufficient to pay for lodging and subsistence with a little left over for personal spending. Needless to say, satisfactory performance is demanded of students because they are able to devote full time to their studies, unhampered by the need of working outside school hours.

Graduation requirements

To be eligible for graduation, each student must complete successfully all of the required course work. In addition, he or she must do a thesis project during the fifth year. This project may be a research paper in one of the specialties studied by the student or on a problem related to the teaching of his specialties.

Final oral examinations in the fields of specialization are given at the end of the fifth year. The state publishes for each subject-matter area a list of topics that a student is expected to know. The student is told that he may be examined in any of the topics. An examining committee consisting of three professors and a recorder is set up. Sets of examination tickets (bilety), consisting of three topics or problems, are made up by examining committees of each institute.

At the time of the examination, the student is asked to select one ticket from the set of tickets placed face down on the table. The student is then given a little time to reflect on how to answer the questions, after which the oral examination commences. At the termination of the examination, which lasts from one-half hour to an hour, the student is marked 5, 4, 3, 2, or 1.

A student who successfully passes all of the course work, the thesis project, and the final state examinations, is awarded a diploma at the annual graduation ceremony. Such graduation automatically grants certification to teach in the specialties in which the student has prepared. A teaching position is guaranteed. A graduate may indicate preference for a particular location or school and insofar as possible these preferences are taken into account in assignments. Needs of the state, however, constitute the overriding consideration in all matters of assignment.

Letter to the editor

Dear Editor:

Mr. Wallace Manheimer in his article "Some Heretical Thoughts from an Orthodox Teacher" in the January, 1960 issue of The Mathematics Teacher warns against false modernism. He mentions in passing *fields*, *rings*, and *ideals*. He alludes to the history of group theory. From physics he mentions pi-mesons. I feel that orthodoxy should be careful. His language shows that he is all too well acquainted with the enemy.

David A. Page
Urbana, Illinois

Careers in mathematics

CONFERENCE BOARD
OF THE MATHEMATICAL SCIENCES,
Washington, D.C.

Employment opportunities for the person trained in mathematics exist in teaching, in industry, and in government.

CHOOSING A CAREER means far more than deciding how to earn a living. If a position involves distasteful work in which you can take no pride, don't bother to ask what the pay is. It won't be enough! If the work will be a satisfaction in itself, then start checking up on the over-all advantages and disadvantages of the career.

TEACHING CAREERS IN MATHEMATICS

Are you cut out to be a teacher? When you see someone bungling a piece of work that you know how to do, do you want to show him how to do it? Do you enjoy explaining mathematics to the classmate that bogged down in it? If so, there's a good chance that you will be happier as a teacher than as anything else.

What are the satisfactions? First, teaching is fun.

Second, a teacher of any good solid subject can feel the satisfaction of knowing that he is making a vital contribution to civilization. In particular, everyone by now realizes that "keeping the world running" calls for more and more scientists, engineers, and mathematicians. And the teaching of mathematics at every level is an absolute necessity for producing the scientists, engineers, and mathematicians that are needed.

Third, there is satisfaction in knowing that you have left a lasting mark—on your community, on your country, perhaps on the world. It is a joy to recognize the spark of interest and ability in a young student and to fan it into full flame. You can take honest pride in knowing that you have helped educate many young people and have given some of them a start toward their careers in science or engineering or mathematics, and that, years later, you will be fresh in the minds of men who have forgotten the names of today's "prominent citizens."

For the man or woman who loves research and combines it with university or college teaching, there are other important satisfactions. Some find in research a joy in tackling tough problems that is a highly magnified version of the fun of solving a puzzle. Others have an itch to find out mathematical facts never known before. Still other students enjoy the creation of interesting and beautiful theorems. These satisfactions are greatest for some persons when the research has no obvious use, while others do not care. Directly useful, or apparently useless but beautiful, it is on the varied results of this century's research that next century's mathematics must depend.

Many scientists and mathematicians have turned down offers of nonacademic positions with high salaries, preferring the academic life. Of those who start a career in teaching, few leave it to change to any other kind of occupation except one, that of housewife.

Are positions open? High schools, col-

leges and universities are now facing a population increase that calls for enlarged faculties in all subjects. But most particularly there will be a shortage of teachers of science and mathematics for a long while to come. A graduate with a good record can count on having his choice of several offers.

High school or college or university? This is in good part a matter of temperament. The high school teacher is likely to be interested primarily in his students. Mathematics teachers in colleges and universities are more specially committed to their subject, and should like to keep themselves fresh and expert in it. The university is the place for the mathematician who wants to put in a considerable part of his time on his own research.

What are the requirements? In most states, a bachelor's degree is enough for high school teaching, but some special courses in education are usually required. You can find the local requirements by asking any school superintendent or professor of education. However, even if a bachelor's degree is enough, a master's degree is better, both in dollars and in a continuing satisfaction from deeper knowledge. For college and university teaching, you should plan to earn a Ph.D.

Here, at the prospect of graduate study, is where many students are frightened when they should not be. You don't need to be an Einstein to get a master's degree —if you've been a good "B" student at a good college, the chances are that you have what it takes. Of course the Ph.D. is further along and requires excellent abilities and hard work; but it is not reserved for supermen. Once, not so long ago, it meant a financial strain on your family to take an advanced degree. Now there is usually enough help available for qualified students to remove this strain. A large percentage of graduate students are completely self-supporting through fellowships and assistantships.

What is the salary? The income of teachers varies so much from place to place that figures may be deceptive. In some states, high school teachers are grossly underpaid; in others, they earn quite comfortable salaries. Any local high school teacher or professor of education can tell you the situation in your state. As of now, a young Ph.D. looking for his first position can expect something like $5,000. Top salaries vary, too. There are a few colleges and universities in which the top is $15,000 or more for the nine-month teaching year.

As a side advantage, teaching offers security. A teacher, once established, has a position until retirement age if he so chooses. Retirement plans are usually good, too.

Industrial and government careers in mathematics

Until recent times, the principal career opportunity for anyone with specialized training in mathematics was in the field of teaching. During the last decade, however, there has been an ever-increasing demand in industry and government for competent people who are well trained in mathematics. This is a result, in large part, of the current technological revolution accompanying the development of nuclear energy, automation, electronics, and computing. In general, there is a very wide range of intellectual activity related to mathematics. This includes the work of technicians who are skilled in applying standardized mathematical techniques to the solution of straightforward problems as well as the investigation of problems of a very profound nature with far-reaching implications.

What is applied mathematics? Much of the mathematics (and mathematical statistics) which is of importance in handling the problems that arise in our attempts to explain and control nature is commonly referred to as "Applied Mathematics," as distinct from "Pure Mathematics." The distinction between these two is, however, not always easy to make. One never knows whether some result in pure mathematics

may provide the necessary tool to cope with some "applied" problem. In general, the difference lies in the motivation of the work rather than in the subject matter. In pure mathematics, a subject is developed for its own intrinsic interest, while the applied mathematician during his investigations usually has in mind some specific problem (or class of problems) arising in one of the sciences. Some of the greatest mathematicians, like Sir Isaac Newton, have been stimulated by the problems of nature to launch far-reaching investigations into mathematics.

What is the nature of the work of the mathematically trained person in industry and government? The recent widespread use of electronic digital computers for scientific calculation as well as for commercial data processing has created a need for many thousands of people who have sound backgrounds in elementary mathematics. They are required to set up the detailed instructions which a computer executes in solving some complex problem. Even in computer applications where no extensive use of mathematics is involved, employers often look for people with mathematical aptitude and training. This is done because it is assumed that such people will also be well able to organize the very complex and intricate procedures which are applied on a high-speed digital computer. Thus, many people with bachelor's degrees in mathematics are employed in commercial data-processing installations and, in general, fill many varied positions—even that of salesman—in our increasingly complex technology.

The professional mathematician is often employed as a member of an industrial research team, working with other scientists and engineers.

"It is because the mathematician is expert in analyzing relations, in distinguishing what is essential from what is superficial in the statement of these relations, and in formulating broad and meaningful problems, that he has come to be an important figure in industrial research teams."* In this capacity, many of the disciplines of mathematics may find application. The somewhat vague term, "Operations Research," has been used to refer to this application of mathematical methods to the solution of diverse problems.

What are the satisfactions? Only someone who thoroughly enjoys meeting the challenge of a mathematical problem should consider working as a professional mathematician in industry or government. Apart from the enjoyment of one's work and the satisfaction of the research worker in adding to man's knowledge, the principal satisfaction is the sense of contributing to the technological (and therefore social) progress of man. Such contributions can be made at all levels. While the creativity and intellectual power of a research mathematician of the first rank are valuable assets, it is not at all necessary to have these rare gifts to make significant contributions. The substantial activity, indicated by the widespread use of digital computers, in the application of mathematical and data-processing methods to industry and government has caused this radical change in the number of opportunities for people without advanced graduate training to do useful mathematical work.

Financially, work in applied mathematics can be rewarding. A person with a bachelor's degree in mathematics and a good academic record can expect to start in industry at more than $5,000 per annum. A person with a Ph.D. would start at a significantly higher salary, above $8,000 per annum. Top salaries vary considerably and depend, in part, on the amount of responsibility assumed by the individual. Someone with a bachelor's degree would, in the normal course of events, expect to increase his salary in the long run to about $10,000 per annum, with op-

* Thorton C. Fry, "Mathematics as a Profession Today in Industry," *The American Mathematical Monthly*, Vol. 63, No. 2 (February 1956), p. 76.

portunities to go substantially above this amount.

What are the requirements? A bachelor's degree with a major in mathematics is the minimum requirement for someone working as a mathematician in industry or government. This should be accompanied by sound basic training in the sciences. More advanced mathematical education is desirable if one intends to become a mathematical consultant on an industrial research team. For many such positions, the master's degree would be adequate, however, the doctor's degree is desirable.

There are many instances of people with advanced degrees assuming positions involving the administration of people or of technical affairs, where, although their formal mathematical training is not used extensively, their acquired judgment in scientific matters resulting from this training is important.

What's new?

BOOKS

SECONDARY

Elements of Algebra. (3rd ed.), Howard Levi. New York: Chelsea Publishing Company, 1960. Cloth, xi+161 pp. $3.25.

COLLEGE

Advanced Complex Calculus, Kenneth S. Miller. New York: Harper and Brothers, 1959. Cloth, viii+240 pp., $5.75.

Analytic Geometry and an Introduction to Calculus, A. Clyde Schock and Bernard S. Warshaw. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Cloth, 165 pp., $3.96.

Boundary Problems in Differential Equations, Rudolph E. Langer (editor). Madison, Wisconsin: The University of Wisconsin Press, 1960. Cloth, x+324 pp., $4.00.

Fundamental Principles of Mathematics, John T. Moore. New York: Rinehart and Company, Inc., 1960. Cloth, xv+630 pp., $7.00.

Real Variables. John M. H. Olmsted. New York: Appleton-Century-Crofts, Inc., 1959. Cloth, xvi+621 pp., $9.00.

MISCELLANEOUS

The Birth of a New Physics, I, Bernard Cohen. New York: Doubleday and Company. 1960. Paper, 200 pp., 95¢.

Crystals and Crystal Growing, Alan Holden and Phyllis Singer. New York: Doubleday and Company, 1960. Paper, 320 pp., $1.45.

High-Speed Math Self-Taught, Lester Meyers, abridged and revised by Maxwell Marke. New York: Pocket Books, Inc., 1959. Paper, xxv+290 pp., 50¢.

The Physics of Television, Donald G. Fink and David M. Lutyens. New York: Doubleday and Company, 1960. Paper, 160 pp., 95¢.

Proceedings of an International Symposium on the Theory of Switching, Part I, Part II. Cambridge, Massachusetts: Harvard University Press, 1959. Cloth, Part I, xl+305 pp., $15.00; Part II, vii+345 pp., $15.00.

Ramanujan, G. H. Hardy. New York: Chelsea Publishing Company, 1940. Cloth, 236 pp., $3.95.

A Source Book in Mathematics, Volume I, David Eugene Smith. New York: Dover Publications, Inc., 1959. Paper, xiii+701 pp., $1.85.

A Source Book in Mathematics, Volume II, David Eugene Smith. New York: Dover Publications, Inc., 1959. Paper xiii+701 pp., $1.85.

String Figures and Other Monographs (3rd ed.), W. W. R. Ball, *et al.* New York: Chelsea Publishing Company, 1960. Cloth, vi+72 pp.; 102 pp.; x+179 pp.; v+136 pp., $3.95.

Waves and the Ear, William A. Van Bergeijk, John R. Pierce, and Edward E. David, Jr. New York: Doubleday and Company, 1960. Paper, 235 pp., 95¢.

The binomial theorem

LURLINE STEWART, *Hinds Junior College, Raymond, Mississippi.*

An example shows how a teacher may enrich the students' learning of the binomial theorem.

SEVERAL WEEKS AGO I was asked to make a talk to an undergraduate science club on the subject of the binomial theorem. When I questioned the reason for the choice of this subject, the student president gave me an answer which, when I thought about it, seemed entirely reasonable. He said that most of the students in the club, composed of physics and chemistry majors, had had only one or two lessons on the binomial theorem and that those lessons dealt chiefly with a mechanical way of expanding a binomial with positive integral exponents, and further, that practically none of the students knew very much about permutations, combinations, and probability. He added that in their sciences they needed to know more about the binomial theorem, particularly how to compute the coefficient of the term containing y^r in the expansion of $(x+y)^n$, how to express these coefficients in terms of combination symbols, how to use the binomial expansion as an approximation, and how to compute n factorial when n is a large number.

It occurs to me that some teachers of second-year high school algebra and of college algebra wishing to add a touch of interest or seeking motivation for learning might use some of the ideas discussed here.

Historically speaking, the discovery of the binomial theorem is credited to Sir Isaac Newton in about 1665, although his work was not entirely original. We have reason to believe that he benefited greatly from the work of John Wallis, to whom goes the credit for regarding the denominators of fractions as powers with negative exponents. For the descending geometric progression $x^2, x, x^0, x^{-1}, x^{-2}, \cdots$, which is the same as

$$x^2, x, x^0, \frac{1}{x}, \frac{1}{x^2}, \cdots,$$

the exponents of his geometric series are in the arithmetic progression 2, 1, 0, -1, $-2, \cdots$. He also used fractional exponents which had been invented long before this but had failed to be generally introduced.[1]

Newton's reasoning gives the development of $(x+y)^n$ whether n be positive or negative, fractional or negative. He completed his work and published it in 1676. He gave no regular proof for his theorem, but verified it by actual multiplication. In a letter that he wrote to James Bernoulli, we find, "The numbers prefixed to the members of the powers arise from continual multiplication, as this: (m is the exponent of the power)

$$1\times\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-4}{4}\cdots,$$

[1] Florian Cajori, *History of Elementary Mathematics* (New York: The Macmillan Company, 1896), pp. 237–240.

which process if m (the exponent) be an integer will (after a certain number of places such as the nature of each power requires) terminate again at 1 as it did begin, but if m be a fraction it will (passing it) run on to negative numbers infinitely."[2] Newton gave illustrations using $(1+c)^{1/2}$, then proceeded to multiply $1+\frac{1}{2}c-\frac{1}{8}c^2+\frac{1}{16}c^3\cdots$ by itself to obtain $1+c$.

Newton is buried in Westminster Abbey, and it is said by some that the binomial formula is engraved on his tomb. There is some reason to doubt the truth of this statement, as the theorem cannot be seen on his tomb at the present time, while the Latin inscriptions are still distinctly readable. The tomb is elevated, and one would have to mount a ladder to read anything written there. Some of the old guidebooks state that there are some mathematical figures on a small scroll held by two youths in front of the half recumbent figure of Newton. Believe what you wish!

The case of positive integral exponents was proved by James Bernoulli by the doctrine of combinations, and a proof for negative and fractional exponents was given by Euler.

It should be mentioned that the beginnings of the binomial theorem were found very early. The Hindus and the Arabs used the expansion of $(a+b)^2$ and $(a+b)^3$ in the extraction of square roots and cube roots. But these were found by actual multiplication, not by any law of expansion. Stifel, one of the greatest German algebraists of the sixteenth century, gave the coefficients for the first eighteen powers. Pascal did the same in his arithmetical triangle. The Chinese had discovered this triangular array some three hundred years earlier but had done nothing with it. Most of us are familiar with this "lazy man's" way of writing the coefficients. The successive rows give the coefficients in the successive positive integral powers of $(x+y)$.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Consider now the expansion of $(x+y)^n$ which is usually written as

$$x^n+nx^{n-1}y+\frac{n(n-1)x^{n-2}y^2}{1\cdot 2}+\frac{n(n-1)(n-2)x^{n-3}y^3}{1\cdot 2\cdot 3}+\frac{n(n-1)(n-2)(n-3)x^{n-4}y^4}{1\cdot 2\cdot 3\cdot 4}+\cdots.$$

In the terms involving y, y^2, y^3, and y^4, we observe special cases of the following facts. Designate by r the exponent of y in any term.

(A) The exponent of x is $n-r$, thus $x^{n-r}y^r$.

(B) In the numerical coefficient, the denominator is $r!$

(C) The numerator of the coefficient is the product of r factors, the first being n and each being one less than the preceding factor, the last factor being $[n-(r-1)]$ or $(n-r+1)$.

When (A), (B), and (C) are combined, they state that the term involving y^r is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+1)x^{n-r}y^r}{r!}.$$

This is called the binomial formula. By the use of this formula, we can write any term without writing the other terms, hence we refer to the above as the formula for the general term.

Here we shall digress for a moment for a brief explanation of some symbolism and some elementary ideas on permutations and combinations. Let us consider the number of distinct arrangements which can be made from all or a part of a number

[2] David Eugene Smith, *A Source Book in Mathematics* (New York: McGraw-Hill Book Company, Inc., 1929), pp. 220–221.

of things. These arrangements are called permutations. If there are n things in the group and we use r of them at a time, then we call this the number of permutations of n things taken r at a time and designate it by ${}_nP_r$. We calculate the number of permutations by the rule which states that if one act can be performed in h ways and if after this or simultaneously with this, another act can be performed in k ways, then the two acts can be performed in hk ways. The rule may be extended to any finite number of acts. We now say that ${}_nP_r$ is $n(n-1)(n-2)\cdots(n+r+1)$.

Now, if we were interested in how many groups of things consisting of r things we could form from n things, we would have what we call combinations, and we write this ${}_nC_r$. There is a close relation between permutations and combinations. Note, however, that while the letters *d*, *e*, and *f* may be arranged in six distinct ways, namely, *def*, *dfe*, *edf*, *efd*, *fde*, and *fed*, each of these arrangements is the same combination of letters. Consequently, to compute the number of combinations, we actually compute the number of permutations and divide by $r!$ to eliminate the number of duplications. Then ${}_nC_r$ becomes

$$\frac{{}_nP_r}{r!}.$$

We may write this as

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}.$$

If we multiply both the numerator and the denominator of this fraction by $(n-r)!$, our formula becomes

$${}_nC_r=\frac{n!}{r!(n-r)!},$$

which is the form that appears in your formulas in your science book. If you will compare the coefficient of the general term, that is, the term involving y^r, with the first form for ${}_nC_r$, we note that we can write the expansion of $(x+y)^n$ as

$${}_nC_0x^n+{}_nC_1x^{n-1}y+{}_nC_2x^{n-2}y^2+{}_nC_3x^{n-3}y^3+\cdots{}_nC_rx^{n-r}y^r+{}_nC_{n-1}xy^{n-1}+{}_nC_ny^n.$$

Let us now recall that we did not prove the binomial formula or formula for the general term. A rigorous proof by induction may be found in almost any standard algebra text. However, there is a rather informal intuitive proof we might mention that uses the idea of combinations. Consider the expansion of $(x+y)^n$. All of the terms of this product may be arrived at by choosing in all possible ways either an x or a y, but not both, from each of the n factors $(x+y)$ and multiplying the selected terms. A term involving y^r then arises from choosing y out of r of the factors and x out of the other $(n-r)$ factors. Now, the number of ways of selecting r letters y out of n factors is ${}_nC_r$, since the order of selection is of no importance; therefore the term $x^{n-r}y^r$ is obtained ${}_nC_r$ times, or, in other words, ${}_nC_r$ is the coefficient of $x^{n-r}y^r$.

The use of ${}_nC_r$ is so important that it is frequently called the rth binomial coefficient in the expansion of $(x+y)^n$, without any mention of the meaning of ${}_nC_r$ as the number of combinations.[3] Sometimes this binomial coefficient is written as $\binom{n}{r}$.

An interesting and useful extension of the binomial theorem leads to certain approximation formulas. For example, suppose we attempt the following expansion, proceeding by the expansion rule:

$$(1+x)^{1/2}=1^{1/2}+\tfrac{1}{2}(1)^{-1/2}x+\frac{\tfrac{1}{2}(-\tfrac{1}{2})(1)^{-3/2}x^2}{2}+\cdots,$$

which when simplified becomes $1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$. As an algebraic equation, this is meaningless because the process on the right never terminates. A meaning, however, is given to such an infinite progression in the calculus when we study "infinite series." For our present discussion, and without proof, we shall use the

[3] William Hart, *College Algebra*, 4th ed. (Atlanta: D. C. Heath and Company, 1953), p. 304.

fact that " · · · for certain values of x, in this particular case $x=1$, the above equality is approximate if the expansion terminates at any point beyond the first few terms, although the more terms we include, the better the approximation."[4] To illustrate,

$$\sqrt{2}=(1+1)^{1/2}=1+\tfrac{1}{2}(1)-\tfrac{1}{8}(1)^2\cdots.$$

Thus $\sqrt{2}$ is approximately 11/8 or about 1.4.

Finally, we are ready to discuss the last of the topics, that involving probability. Frequently it is required to compute the probability of the occurrence of an event in n trials when the probability of the occurrence of that event in one single trial is known. In reading the literature on probability, we find that most of the illustrations used, and most of the examples given, deal with throwing a die several times in succession. Perhaps there is some reason for an example of this type. Be that as it may, we shall use an example which occurs in many books, one that requires the finding of the probability of throwing exactly one ace in six throws of a single die. This particular discussion is given by Sokolnikoff.[5] The mutually exclusive cases are as follows:

(1) An ace on the first throw and none on the other five.
(2) No ace on the first throw, an ace on the second, and no aces on the remaining four throws.
(3) No ace on the first two throws, an ace on the third, and no aces on the last three throws.
(4) No aces on the first three throws, an ace on the fourth, and no aces on the last two throws.
(5) No aces on the first four throws, an ace on the fifth, and no ace on the last throw.
(6) No aces on the first five throws, and an ace on the last throw.

The probability that an event will occur in a particular set of r trials and fail in the remaining $(n-r)$ trials is $p^r(1-p)^{n-r}$.

The probability of the first occurrence is $1/6(5/6)^5$, since the probability of throwing an ace on the first throw is 1/6, and the probability that it will not show on the succeeding five throws is $(5/6)^5$. Likewise, the probability in (2) is $5/6(1/6)(5/6)^4$, and in (3) it is $(5/6)(5/6)(1/6)(5/6)^3$, etc. It is clear that the probability of any one of the six combinations is $1/6(5/6)^5$. Since the cases are mutually exclusive, the probability that some one of the six combinations will occur is $p=6(1/6)(5/6)^5$.

It should be observed that the probability of obtaining any of the above combinations is always the same, so that to obtain the probability of occurrence of one of the set of mutually exclusive events, all that is necessary is to multiply the probability of the occurrence of any specified combination by the number of distinct ways in which the events may occur. This leads to the formulation of an important theorem which is frequently called the binomial law.

The binomial law states that if the probability of occurrence of an event in a single trial is p, then the probability that it will occur exactly r times in n independent trials is $p_r={}_nC_r p^r(1-p)^{n-r}$ where

$${}_nC_r=\frac{n!}{r!(n-r)!}.$$

I believe that the method of proof of this theorem is obvious from the discussion of the specific case mentioned above.

The binomial law on which a large portion of probability is based is exact, but it is too complicated for purposes of computation. The labor of computing the values of the factorials when n is a large number is burdensome, to say the least. It is for this reason that we wish to develop an approximation for $n!$ when n is large.

Such a formula, which furnishes a good approximation to $n!$, was developed by Stirling. In this formula, the percentage of error made by using the formula as an approximation to $n!$ is small when n is sufficiently large, whereas the actual error itself increases with the increase in n. It is

[4] Israel H. Rose, *A Modern Introduction to College Mathematics* (New York: John Wiley and Sons, Inc., 1959), p. 365.

[5] I. S. and E. S. Sokolnikoff, *Higher Mathematics for Engineers and Physicists*, 2nd ed. (New York: McGraw-Hill Book Company, Inc., 1941), pp. 501–502.

claimed that for n larger than 10, the error is less than 1 per cent. Stirling's formula states that $n! \simeq n^n e^{-n}\sqrt{2\pi n}$. The computation in even this formula becomes tedious, but by using $e=2.718$ and $\sqrt{2\pi}=2.507$, the work is lessened considerably.

One of the approximations is known as the Poisson formula, or the law of small numbers. According to Sokolnikoff,[6] the wide range of applicability of this law can be inferred from the fact that it has been used successfully in dealing with such problems as those of beta-ray emission and the expected sale of commodities. This author gives a proof for Stirling's law and also derives the law of small numbers.

Let us now apply Stirling's formula to our probability formula:

$$p_r = \frac{n!}{r!(n-r)!}\, p^r(1-p)^{n-r}.$$

Replacing $n!$ and $(n-r)!$ this becomes

$$p_r = \frac{n^n e^{-n}\sqrt{2\pi n}}{r!(n-r)^{n-r}e^{-(n-r)}\sqrt{2\pi(n-r)}}\, p^r(1-p)^{n-r}$$

which simplifies into

$$\frac{n^r e^{-r}}{r!\left(1-\frac{r}{n}\right)^{n-r+1/2}}\, p^r(1-p)^{n-r}.$$

Since, by hypothesis, r is small compared with n,

$$\left(1-\frac{r}{n}\right)^{n-r+1/2}$$

is almost equal to

$$\left(1-\frac{r}{n}\right)^{n},$$

which for large values of n differs little from e^{-r}. By definition,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

is e.

Similarly, $(1-p)^{n-r}$ is approximately equal to $(1-p)^n$, which in turn is nearly equal to e^{-np}, since

$$(1-p)^n = 1-np+\frac{n(n-1)p^2}{2!}-\cdots,$$

and

$$e^{-np} = 1-np+\frac{n^2p^2}{2!}-\cdots.$$

Substitution of e^{-r} for

$$\left(1-\frac{r}{n}\right)^{n-r+1/2}$$

and e^{-np} for $(1-p)^{n-r}$ leads to the desired law of small numbers:

$$p_r = \frac{(np)^r e^{-np}}{r!}.$$

By using this law, our computations will be lessened considerably.

As indicated in the beginning, this study is not intended as an exhaustive one nor as a learned one. Rather, it is supplemental and, in some cases, remedial in nature. Perhaps there are those among us who have gotten into a rut teaching these topics and need a small reminder that they are more important than the space devoted to them in the average algebra text would seem to indicate. Certainly if we teach these topics thoroughly, our science majors will rise and call us blessed!

[6] *Ibid.*, pp. 509–512.

Curve fitting

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Given the graph and data, find the equation that would have this graph.

OCCASIONALLY one encounters problems in which a portion of a curve is given and it is required to find the equation of the curve to extrapolate it. These problems may range from restoring pottery from the grave of some long-forgotten Indian or mending a casting for damaged machinery, to determining whether the path of a comet, briefly visiting our solar system, is a parabola or an ellipse. For the most part we will deal, not with the applications, but with the methods of curve fitting.

Curve fitting is essentially a trial-and-error process, and it is as fascinating as a jigsaw puzzle. Only a few of the simpler curves will be discussed here, since a large volume could not describe them all. Some require very high-powered mathematical theory. The curves we shall discuss will be circle, parabola, hyperbola, ellipse, and a form of the logarithmic curve.

The first step is to choose the axes. While an equation may be written to fit the curve with respect to any set of axes, a felicitous choice of a pair of axes may result in a greatly simplified expression. Only experience with curves can guarantee a wise choice of axes, but there are some guides to help the inexperienced. If an axis of symmetry can be determined graphically, then choosing this axis is usually wise. Locating critical points of the curve or asymptotes can give a clue to the best choice of axes.

We are given the points in the table below:

Point	x	y
A	6.4	8.0
B	7.0	7.9
C	8.0	7.5
D	9.0	6.7
E	10.0	4.5

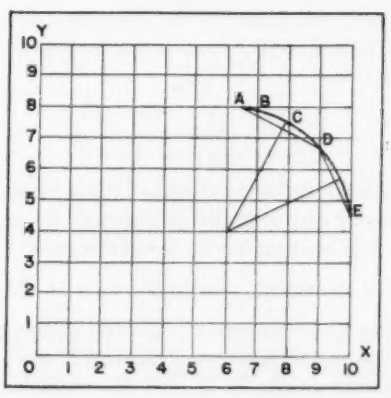
These points are plotted in Figure 1. In this case, we may as well take the axes given. It would appear to the eye that this curve is a circle. The equation for a circle is

$$(x-a)^2+(y-b)^2=R^2,$$

where a and b are the co-ordinates of the center and R is the radius of the circle.

We could substitute the points in the equation and solve for a, b, and R. There are three unknowns, so three points are needed. It should be noted here that a circle may be drawn through any three points, so at least *four* are needed to define a curve. We are given five points, so the curve is defined. In this case, however, it is simpler to find a, b, and R graphically. Recalling from elementary geometry that the perpendicular bisectors of the chords of a circle intersect at the center, we draw chords AD and DE and construct the perpendicular bisectors of the chords. By measurement they intersect at (6, 4), and R is found to be 4. It is then easily deter-

Figure 1



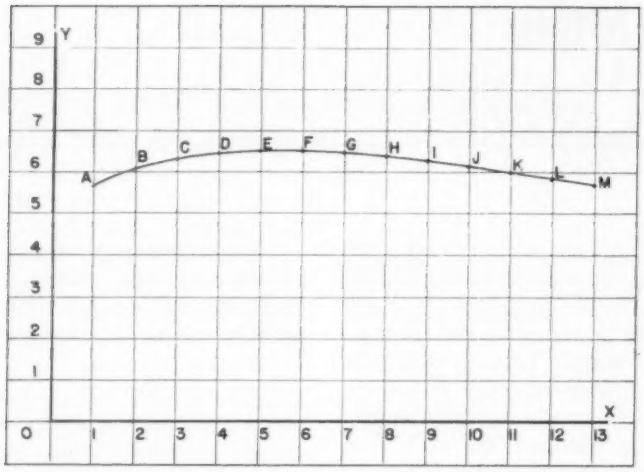


Figure 2

mined that points B and C fall on a curve with $R=4$ and center at (6, 4). The curve is a circle, and the complete curve may be drawn.

In the next case, the curve is not so obvious. Here in Figure 2 is a curve in which no axis of symmetry is obvious and the curve is definitely not a circle. We might try any one of a number of curves before we find one that fits. Let us try a parabola. The general equation for a parabola is

$$y=ax^2+bx+c.$$

There are three unknowns, a, b, and c, to be determined. Hence three points are needed. Substituting the co-ordinates of points A, F, and J from the curve into the equation:

A gives $a+b+c=5.7$
F gives $36a+6b+c=6.5$
J gives $100a+10b+c=6.2$

Solving these simultaneous linear equations by any appropriate method, we get the following values for the coefficients:

$$a=-0.026, \quad b=0.34, \quad c=5.34.$$

The question now is, will the equation fit the rest of the curve? Putting it to the test with points D, K, and M, we find:

Point	*Actual value*	*From the equation*
D	6.4	6.3
K	6.0	5.9
M	5.7	5.4

We must decide if this is a sufficiently close fit. That would depend on the purpose of the investigation. Perhaps this is a good enough fit; if not, there are two courses open. We might try other equations, or we might carry out the work with greater accuracy. Our accuracy is limited by the graph, and the search for other curves might be long and laborious. Another possibility might be to treat the curve as a compound curve, using one equation from A through F and another for the rest of the curve.

Figure 3 would seem to be a hyperbola. If so, then it may be placed in a standard position with the x and y axes as asymptotes. After some trial and error, the curve may be placed on the axes as shown in Figure 4. The equation for a hyperbola in this position is

$$xy=c.$$

Substituting in the following points, we get these values for xy,

Figure 3

x	y	xy
1.0	4.4	4.4
2.0	2.4	4.8
3.0	1.6	4.8
4.0	1.2	4.8
5.0	0.9	4.5
6.0	0.7	4.2

Once again we see that the equation fits the curve very well for the midportion, but fits more poorly the further out we go. A better fit might be achieved by locating the axes more accurately.

A problem in curve fitting which is of great importance in astronomy is to determine from a few observations whether a comet follows the path of an ellipse or of a parabola. If it follows an ellipse, it is most likely to be an old friend returning to visit our regions once more. If the comet follows a parabola, it is a visitor from outer space and will return no more. The complete theory is very complicated and beyond the scope of this article. However, we can gain some insight into the basic principles by taking an example of an ellipse or parabola in polar co-ordinates. When the focus is at the origin and the major axis is horizontal, the equation for an ellipse, parabola, or hyperbola is*

$$\rho = ep/(1 - e\cos\theta),$$

* There are actually four equations that must be tried, depending on the position of the curve. For a fuller discussion, see Fuller, *Analytic Geometry* (Reading, Massachusetts: Addison-Wesley Publishing Co., 1954), p. 128.

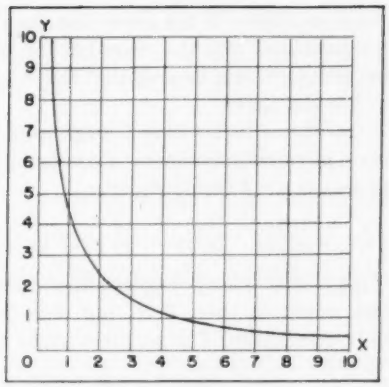


Figure 4

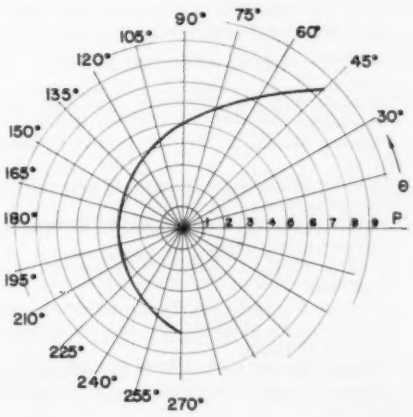
where ρ and θ are the co-ordinates, e is the eccentricity, and p is a constant depending on the geometry of the curve.

Figure 5 shows a portion of a curve we want to test. The points given are:

θ	45°	90°	180°	270°
ρ	9.48	5.00	3.00	5.00

Since the cosine 90° is 0, if we substitute the values for $\theta = 90°$ into the equation, we find that $ep = 5.00$. Substituting ep and any of the other points gives us $e = \frac{3}{4}$ and $p = 7.46$. The eccentricity of an ellipse is always less than 1, while that of a parabola is 1, and that of a hyperbola is greater than 1. Hence, since $e = \frac{3}{4}$, our

Figure 5



curve is an ellipse. If the values for e and p are substituted into the equation for the other points, it will be seen that the equation fits the curve.

Logarithmic forms play a large part in certain physical phenomena. Data from an experiment are tabulated below:

x	1.0	1.5	2.0	2.5
y	5.00	11.17	25.0	55.90

There are many logarithmic forms which might fit these data, but only two will be considered. The simplest might be,

$$y = x^n$$

or

$$\log y = n \log x.$$

In the logarithmic form, this is a linear equation, and plotting $\log y$ versus $\log x$ should result in a straight line. This is usually done on special paper called log-log paper where the values are plotted directly and the divisions on the paper are spaced logarithmically. The value of n is then the slope of the line. Plotting $\log y$ versus $\log x$ does not yield a straight line in this case.

A second form to try is

$$y = a^x$$

or

$$\log y = x \log a.$$

Since a is a constant, $\log a$ is a constant, and we see that the equation is logarithmic on one side and linear on the other. Solving for $\log a$,

$$\log a = (1/x) \log y.$$

Substituting values from the table of data gives $\log a = 0.699$ and a equal to 5. This could be done by plotting on semi-log paper, which has one axis logarithmic and the other linear. We should get a straight line with a slope of $\log a$.

Many more examples could be given. The subject of curve fitting is limitless; this should, however, give the reader an insight into the field and enable him or her to carry on where this article leaves off.

Have you read?

Flaun, Lawrence S. "A Credo For Teachers," *Journal of Teacher Education*, December 1959, pp. 422–424.

This article is not mathematics, but we are all teachers. Do you have a "Credo?" If so, you will want to read this for comparison. If you don't have a credo, you will want to know what should be included. The author says teaching is a calling and one enters it because one believes:

1. Children should have opportunities for growth.
2. Children can be improved mentally and physically.
3. Children should be helped to know themselves.
4. Children should learn to control and modify their activities.
5. Children should be challenged to investigate the uncharted ways.
6. Children should be helped to free themselves from fear.
7. Children should be taught to understand the magic quality of ideas and not be satisfied with the obvious.
8. Children should be helped in developing the ability to make free choices and adequate adjustment.

There are other items you will want to read, but it all sums up to a belief that the child is a world in the making. Think about this.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Gardner, Martin. "Mathematical Games," *Scientific American*, September 1959, pp. 236–244.

Today it is popular to have a collecting habit and this article tells about the "Puzzle Collection" of Lester A. Grimes, a retired fire protection engineer. He has over two thousand puzzles. The article gives some interesting information about puzzles. The "Tangram," a Chinese puzzle of several thousand years ago, became popular in Naploeon's era. The directions for one tangram are given, and you or your students can take off from there. All you need is a piece of cardboard and scissors. A second puzzle with pegs is explained that was popular in Victorian England. Who thinks up all these, and what is the mathematics involved? Read this and you will be surprised.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Values of mathematics for the modern world[1]

HERMAN ROSENBERG, *New York University, New York, New York, and Jersey City State College, Jersey City, New Jersey.*

What are the values that a student derives from the study of mathematics?

INTRODUCTION: TWENTIETH-CENTURY CHALLENGES

THE EMERGING ROLE of mathematics in American education has been dramatized by at least two major explosive developments of our time:

1. our *national* economic growth in the new *Age of Automation.*
2. our *international* rivalry in the new *Space Age.*[2]

Thus, the current zooming demand for greater emphasis on mathematics education arises from at least two basic needs of modern war and peace:

1. the need for personnel with adequate mastery of mathematics to design or to operate the "fabulous" new automatic computing machines that are becoming increasingly vital in our economic expansion.
2. the need for personnel with adequate mastery of mathematics to design or to operate the vehicles needed in the exploration of space—an exploration deemed necessary for reasons such as national security. Indeed, it has been predicted that the next world war will be "merely" a battle of mathematicians.

The purpose of this article is to examine the potential values of mathematics education in the light of the total needs of modern education. There are, for example, strong current pressures to fashion mathematical programs in various directions. Through a broad examination of the total values of mathematics it may be possible to gain considerable perspective that may permit improvement in the evaluation and/or modification of such mathematics programs.

Hence, the remainder of this article is organized in this way: first, the major potential values of mathematics education will be identified; second, the article will be concluded by a brief discussion of the evaluation and implementation of such values.

THE VALUE OF MATHEMATICS EDUCATION AS SPECIALIZATION EDUCATION

In a discussion of values of mathematics, it may be helpful to distinguish between values of mathematics in *general* education and values of mathematics in *specialization* education. The latter values will be considered first, for they are easily identified.

[1] Inspired, in part, by a panel discussion, "Potential Values of Mathematics and Science in the Modern Age," presented at the 1959 Annual Spring Conference of the Alumni and Faculty of the School of Education of New York University. The chairman of the panel was Dr. John J. Kinsella (Professor of Education at New York University), and the writer served as cochairman and moderator. The panel members were Dr. Elwood J. Winters (Associate Professor of Education at New York University), Dr. Edward J. Zoll (Mathematics Co-ordinator at the Jonas Salk School, Levittown, New York), Miss Beryl Hunte (mathematics teacher at Friends Seminary, New York City), and Miss Dorothy Geddes (teacher at Hunter College High School, New York City, and teacher-broadcaster for the Regents Educational TV Project, WPIX).

[2] For a recent interpretation of the effects of such international space rivalry on American mathematics education, see, for example, Herman Rosenberg, "The Real Menace of the Sputniks to Mathematics Education," *School Science and Mathematics*, LIX (December 1959), 723–730.

Thus, it is not difficult to recognize the role of mathematics education in providing a vocational *tool* for careers[3] involving relatively large amounts of mathematics. For example, the detailed study of mathematics is of fundamental importance to persons preparing for such careers as those of engineers, scientists, statisticians, actuaries, mathematicians in industry and/or research. Some occupations may be regarded as less glamorous in the sense that the mathematics needed is reduced in amount and difficulty. These occupations include some in industry (machinists, carpenters, and electricians, for example), commerce (insurance agents, bookkeepers, accountants), and in other areas (farmers, surveyors' assistants, supporting technicians). Indeed, it is not difficult to develop the thesis that mathematics may be a fundamental tool in preparing for numerous careers in political, economic, and social areas of modern life.

Some insight into the reasons for the utility of mathematics in so many careers in our modern age may be derived from the fact that the strength of applied mathematics lies in its powers of both description and prediction. For example, mathematics may eliminate the need for the so-called slave labor of actual direct measurement (as in the case of indirect mathematical prediction of heights of buildings). Mathematics permits prediction of distances not possible to measure directly even if "slave labor" were available (as in the case of indirect mathematical prediction of distances across inaccessible rivers). Consider, too, the strategic role of mathematics in predicting distances between two planets, the speeds of moving bodies, maximum profits, minimum amounts of time needed, economical space utilization, probable success or failure in certain careers, growth of population and money, and the flow of floods of water and of rumors.

Langer has recently emphasized the dependence of scientific evolution on mathematics. To illustrate his thesis that the great technological achievements of our time have been achieved mathematically, he cites such achievements as flight transportation " . . . through the atmosphere or under the surface of the sea, the mass communication mediums of radio and television, the recording of music, atomic fission and fusion, radar detection and rocket power, the analysis of spatial radiations, the expanded knowledge of the astronomical universe and of the constitution of the earth . . . "[4]

Values of Mathematics Education as General Education

What are the major potential values of mathematics education in *general* education? At least two groups of such potential values may be considered: incidental, utilitarian-vocational values; other values (particularly cultural and disciplinary), to be treated later in this article.

The story is told of a prize fighter who boasted that his manager had offered him a fourth of the gate and continued, "But I held out until he gave me a fifth. I believe if I had stuck to it, I could have got-

[3] See the final report of the Commission on Post-War Plans, *Guidance Pamphlet in Mathematics For High School Students* (Washington, D. C.: The National Council of Teachers of Mathematics, 1947).

[4] Rudolph E. Langer, "To Hold, as 'Twere, the Mirror up to Nature; to Show the Very Age and Body of the Time," The Mathematics Teacher, LII, (December 1959), p. 595. For an interesting article on the rather surprising use of mathematics (the law of cosines of trigonometry) in medicine (plastic surgery), see Laura Guggenbuhl, "An Unusual Application of A Simple Geometric Principle," The Mathematics Teacher, L (May, 1957), pp. 322–324. (Reading the latter article is not expected to leave any scars!) For a discussion of the relation of mathematics to reproduction and genetics as well as plant leaves, seashells, and religion, see Herman Rosenberg, "Higher Dividends with Series of Higher Orders," *The New Jersey Mathematics Teacher*, VIII (May 1952), pp. 10–17. For the role of mathematics as a tool in illuminating the analysis of the growth of love, the growth of population, and the relationship between love and population, see Herman Rosenberg, "Modern Applications of Exponential and Logarithmic Functions," *School Science and Mathematics*, LX (Spring 1960).

ten a sixth."[5] The story illustrates the possibility that even in careers that are seemingly nonmathematical some incidental knowledge of mathematics may be quite a useful tool.

Incidental utilitarian-vocational values of mathematics are possible for such persons as the general citizen, consumer, and homemaker. Here, important, but relatively less spectacular, types of mathematics may be useful in areas such as budgeting, adjusting recipes, computing income taxes, and interpreting newspaper statistical data (on population, employment, stocks, or crops). Perhaps it may now be clear why some women have felt that men are such calculating animals!

Cultural values of mathematics education

To what extent may mathematics provide cultural values for students in our modern age? The answer may be found in at least two possible directions.

Mathematics may be regarded as a significant aspect of our culture to be enjoyed for its *own* sake. Thus, mathematics may be comparable to music in that mathematics may be, for some people, a creative *art* or craft. One may actively practice mathematics, just as a musician practices the playing of music; or one may passively appreciate mathematics, just as an auditor of music (or viewer of art or reader of literature) appreciates another aspect of our culture. (Indeed, there are persons who see mathematics as a suitable competitive sport, so that students may be not only athletes but also "mathletes"!) When mathematics is regarded as a pleasurable aspect of our culture, the basic *satisfactions* possible to the culture-seeker may be aesthetic, for mathematical figures may possess symmetry, proportion, balance, and rhythm; intellectual, for mathematics may offer intellectual play or challenge; or even emotional, for student discovery of mathematical principles is frequently accompanied by open displays of joy.[6]

Mathematics may be an aid in the appreciation or learning of *other* aspects of our culture—aspects such as art, literature, and philosophy. Here, it may be helpful to think of mathematics in two ways, as a language and as a science.

Consider mathematics as a modern *language*. Just as one may communicate (write, read, speak, or listen) in the English language, one may, for communication purposes, utilize the terminology or symbolism of mathematics. Simple mathematical terms (such as triangle, circle, percentage, and average) and mathematical symbols (such as those of graphs and tables) appear constantly in nonmathematical books, magazines, comic strips, television and radio programs, drama, films, etc. The material presented in such vehicles for communication may be better appreciated if the mathematical language employed is clearly understood.[7] Indeed, an ecologist declared recently that such terms as correlation coefficient, confidence limits, and standard error are used so

[5] Lucien B. Kinney and C. Richard Purdy, *Teaching Mathematics in the Secondary School* (New York: Rinehart and Co., 1954), p. 6.

[6] The story is told of the discovery made by the great mathematical genius of antiquity, Archimedes. His joy in making the discovery was so great that, jumping from his bath, he forgot to dress before his open display of joy in the city streets!

Morris Kline, in *Mathematics in Western Culture* (New York: Oxford University Press, 1953), p. ix, has written, "As an incomparably fine human achievement, mathematics offers satisfactions and aesthetic values at least equal to those offered by any other branch of our culture."

The Commission on Mathematics of the College Entrance Examination Board has recently declared, "Mathematics is eminently worthy of study in its own right: it is a vital and shining example of mankind's creativity, one of the great cultural achievements of the ages. Few subjects can rival its ability to stimulate the inquiring mind. Every educated man and woman should have an opportunity to know at first hand something of the intellectual excitement and deep satisfaction that mathematics can offer." The quotation is from *Program for College-Preparatory Mathematics* (Princeton N. J.: Educational Testing Service, 1959), pp. 12–13.

[7] It would be nice, too, for all writers in literary circles to understand the mathematical language they use. For example, this mathematical faux pas from a recent textbook, "By far the greater half . . . ," is cited by Kinney and Purdy in *Teaching Mathematics in the Secondary School* (New York: Rinehart and Co., 1954), p. 6.

often in " · · · biological writing that ignorance of them almost amounts to illiteracy."[8]

Simple examples of the use of mathematical language in general culture are easy to locate. Thus, in the film "The Bachelor and the Bobby-Soxer," Judge Myrna Loy reprimands her teen-aged daughter for her action in a high school class in defining a triangle as "two men in love with the same woman." In the film "She Wrote the Book," Joan Davis suggests to her friend that his engineering problem of building a bridge may be simplified by the use (from trigonometry) of the law of tangents, and her lost memory is regained by the sight of an "improper" calculus symbol. Dagwood is quite "flabbergasted" by his young son's wise suggestion that a shelf may be easily built by using the Pythagorean theorem from geometry. Leonard Bernstein, in a recent television program, utilized the mathematical language of exponents to convey to his audience something of the infinite variety of ways in which the tune "How Dry I Am" can be played.

Now think of mathematics as a *science*. The facts or principles in the mathematical body of scientific knowledge may be applied directly in the appreciation or learning of such cultural fields as the arts and the social sciences. For example, the appreciation of the design of stained glass windows, architecture, or paintings may be increased by the knowledge of the use therein of mathematical principles such as that of dynamic symmetry. Thus, Renaissance artists such as Raphael, Michaelangelo, and Da Vinci deliberately based their art on geometric principles. In his painting of the Mona Lisa, Da Vinci obtained the famous smile by tilting the lips so that the ends of the lips lie on a *circle* that touches the outer corner of the eyes. To improve understanding of Da Vinci's painting of the Last Supper, it may be helpful to note at least three mathematical aspects: the visible part of Christ conforms to the sides of an equilateral *triangle;* the head of Christ is emphasized by the use of mathematical *perspective;* the face of Christ is emphasized as the center of interest by using the intersection of the diagonals of a *rectangle.*[9]

[8] Edwin P. Martin, "The Mathematics Market," THE MATHEMATICS TEACHER, LII (December, 1959), p. 617.

DISCIPLINARY VALUES OF MATHEMATICS EDUCATION

If Will Rogers were alive today, he might comment that "the way the do-it-yourself fad is spreading, it may not be long before people think for themselves." Mathematics teachers have long hoped that the study of mathematics may assist in some way in the improvement of the ability to think correctly and possibly even creatively.

Indeed, there was a time when the justification for the study of mathematics rested rather shakily on the theory of an automatic transfer of training from mathematics to nonmathematical endeavors.[10]

[9] Some insight into the basis of the potential cultural values of mathematics may be obtained from Lancelot Hogben, *Mathematics for the Million* (New York: W. W. Norton, 1937). Hogben's thesis that mathematics has been the mirror of our civilization has been more fully developed recently by Morris Kline (in *Mathematics in Western Culture*), who suggests that mathematics is a prime molder and a major element of our culture. He states, "Mathematics has determined the direction and content of much philosophic thought, has destroyed and rebuilt religious doctrines, has supplied substance to economic and political theories, has fashioned major painting, music, architectural, and literary styles, has fathered our logic, and has furnished the best answers we have to fundamental questions about the nature of man and the universe" (*op. cit.*, p. ix). To substantiate this statement, Kline examines the history of Western civilization. He considers, for example, mathematical bases for the work of such men as Descartes, Copernicus, Kepler, Adam Smith, Malthus, and John Locke.

[10] It may be interesting to read the charge made at the turn of the century by the novelist W. J. Locke, "I earned my living at school slavery, teaching to children the most useless, the most disastrous, the most soul-cramping branch of knowledge wherewith pedagogues in their . . . folly have crippled the minds and blasted the lives of thousands of their fellow creatures—elementary mathematics . . . it teaches boys to think, they say. It doesn't. . . . Its sacrosanctity saves educationalists an enormous amount of trouble. . . . " The quotation is from David Eugene Smith, *The Teaching of Geometry* (Boston: Ginn and Company, 1911), p. 13.

The twentieth-century overthrow of the *automatic* aspect of the transfer theory and the recognition in recent decades of the possibility of transfer of training under planned conditions have had significant effects on the development of modern methodology in the teaching of mathematics. Such doctoral studies as those of Harold Fawcett,[11] Gilbert Ulmer,[12] Richard Gadske,[13] and Harry Lewis[14] have explored the possibility of utilizing mathematics in the development and/or improvement of critical and creative thinking. Thus, considerable emphasis today in the teaching of mathematics is placed on the method of student understanding and discovery of mathematics, opportunities for original (creative) proofs of mathematical relationships inductively discovered, recognition of mathematical and similar nonmathematical structures, and applications of mathematical reasoning in nonmathematical situations of daily living. In brief, the value of mathematics as a *method of thinking* is being given increasing attention.[15]

Other disciplinary values have been claimed for mathematical study. They include the possible improvement of other powers (such as memory and spatial imagination), ideals and habits (such as persistence, accuracy, honesty, and orderliness), and attitudes and appreciations (self-confidence, preference for reason over unreliable authority). Only as recently as 1952, however, the doctoral study of Ernest Ranucci indicated that at least one of these powers, spatial imagination, is not automatically improved by the study of mathematics (solid geometry).[16]

Concluding remarks: evaluation and implementation of the potential values of mathematics education

Which one of the potential values of mathematics is the most important value? The answer is that the relative importance of a potential value of mathematics education may depend on a number of factors. Such factors include:

The nature and purpose of the education (general versus special). The student seeking a career involving considerable uses of mathematics has needs very different from the terminal needs of students in general education.

The level of education. The elementary school child encountering his first experiences in mathematics education and the college student majoring in mathematics have, obviously, very different mathematical outlooks.

The nature of the learner. The slow learner of mathematics and the rapid learner of mathematics differ widely in their needs—fundamental and creative.

[11] *The Nature of Proof*, Thirteenth Year book of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1938).

[12] "Teaching Geometry to Cultivate Critical Thinking" (Unpublished doctoral dissertation, University of Kansas, 1939).

[13] "Demonstrative Geometry as a Means of Cultivating Critical Thinking" (Unpublished doctoral dissertation, Northwestern University, 1940).

[14] "An Experiment in Developing Critical Thinking Through Teaching of Plane Demonstrative Geometry" (Unpublished doctoral dissertation, New York University, 1950).

[15] Support for such an emphasis in mathematics has been provided by such groups as the President's Commission on Higher Education. The commission wrote in *Higher Education for American Democracy* (New York: Harper and Brothers, 1947), I, 52: "That the student grasp the processes involved in scientific thought and understand the principles of scientific method is even more important than that he should know the data of the sciences." The commission indicated its view that mathematics can be made to contribute richly to the development of the power to think, and it declared on page 57, "Development of the reasoning faculty . . . should be the constant aim of all education, in every field and at every level. . . . Insofar as education is not indoctrination, it is discovery."

[16] "Effect of the Study of Solid Geometry on Certain Aspects of Space Perception Abilities" (Unpublished doctoral dissertation, Columbia University). However, it is well to recall the remarks made by Charles Butler and F. L. Wren, "There is no longer any doubt that transfer does take place. . . . The most competent evidence indicates that, in order to achieve the disciplinary values of any subject, it is necessary to teach that subject with that specific purpose in view," *The Teaching of Secondary Mathematics* (New York: McGraw-Hill Book Company, 1951), p. 82.

The nature of the times. The emphasis on certain values of mathematics has frequently been affected by the existence of peace and "running hot and cold wars" (or by the existence of unusual periods of economic growth or depression). Thus, during the Second World War, military and industrial applications of mathematics were given considerable emphasis. During the depression of the 1930's, there was a tendency to downgrade the importance of mathematics in the curriculum. The recent cold war between the United States and Russia, made more crucial by the creation of Russian sputniks and luniks, has led to a strong demand for more mathematics and more difficult mathematics for our students to meet the challenge of Russian education. Some appreciation of the connection between our curricular values and our times may be derived from the observation that, within the same decade of the 1950's, our American schools were accused of being too Russian and not Russian enough. Thus, there is a paramount need for balance in ordering the potential values of mathematics education, a need, for example, for not ignoring potential cultural and disciplinary values in general education in times of emphasis on more immediate vocational values.

Throughout this article, the values of mathematics education have been called potential values. Today, it is increasingly being recognized that the achievement of the values claimed for mathematics education is not automatic. The attainment of the potential values of mathematics education depends on such factors as:

1. the content and organization of the total curriculum, of which mathematics is a related part;
2. the materials of instruction permitting achievement of such values;
3. the existence of teachers of mathematics well prepared and desirous of realizing the potential values of mathematics;
4. the support of an enlightened school administration and community desirous of making possible those values of mathematics deemed best for society and each individual in our modern age

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Have you read?

Gobler, Leonard H. "Some Antics with Semantics," *School Science and Mathematics*, January 1960, pp. 14–16.

Communication is difficult to say the least, as this short dialogue between the layman and mathematician so clearly shows. It deals with such a simple concept as the one that the number of all numbers is equal to the number of even numbers. A theater analogy is used, so are heads and human ears, and it all ends where "equals" and "twice-as-many" may mean the same. But you better read it—I am confused.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Gray, Nelson B. "All Number Systems Have Base 10," *The Duodecimal Bulletin*, December 1959 (1173), pp. 43–49.

The nature of the number system becomes meaningful only when we have the opportunity to make comparisons with other number systems. This article does just that from binary to duodecimal, from counting to powers of 0 to 4, from factorability to simplicity of counting.

This article also gives a good exposition on converting from one base directly to another through a simple process using only the base, the digits, multiplication, and addition. There is no raising to powers or using large divisors. Read this article. It is good, but don't ignore the bias.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Providing for the basic student in the junior high school

WILLIAM G. MEHL, *Wilson Junior High School, Pasadena, California.*

Some helpful suggestions are presented for working with the special student considered in this article.

AMERICAN EDUCATION is unique in its attempt to provide for all children who have a potential for some kind of learning. American teachers are therefore often faced with undertaking to provide adequate instruction and materials for the gifted, the average, and the slow learner. It is the last, referred to here as the *basic* student, which gave impetus to the writing of this paper.

In general, the basic student is one who does not achieve as rapidly or as thoroughly as the average student. Indeed, he may be as much as two or three years behind in as many as all subjects, and may, most usually, display a definite lag in mathematics. Typically, his reasoning ability will test lower than his ability to compute. This may or may not be indicative of previous rote training to the point of mechanical mastery without any veritable understanding of the subject. It is also quite probable that the extremely basic student will tend to terminate his formal education at his first legal opportunity, and most certainly at the completion of high school.

During the years we have the basic student in our midst, we are obligated to provide him with any and all mathematical instruction commensurable with his ability and future good citizenship. However, such an effort is not without special problems that tend to arise due to the very nature of the specialities of the instructional climate.

One of the problems confronting teachers of basic students is that of providing adequate text material. Junior high school students, however lagging in their pursuit of studies, do not relish the usual remedial approach to the study of mathematics. While remedial procedure may be indicated, too often the teacher resorts to handling the situation by supplying texts which are one, two, or more years behind the actual grade level of the student. This action may add to the frustration already encountered by the student, as well as lower his prestige or status among his peers. In addition, the regression and repetition are often embarrassing to the student, not to mention that they are probably irritating, degrading, and boring.

Another problem which may arise in the teaching of basic students is that of increased abnormal behavior. Some of the basic students will be considerably more mature physically than others because a wide chronological age difference is more frequent in basic than in other groups. Attention spans will vary extensively, and it is not unusual to find some of the students refusing to do any constructive work at all. Homework will often not be taken home. A few students will not trouble to bring books, pencils, nor paper to class. Quite reasonably, it is easy to understand how some teachers might become exasperated with a constant repetition of such behavior. It is frustrating to the sincere teacher who wants to share his special techniques and mathematical knowledge with his students.

The last problem of a serious nature, and perhaps most worthy of mention, is

the frequent noncommittal attitude of the parents of the basic student. Very often these parents are *not* members of the P.T.A. or other groups interested in providing for school children. The parents may *never* visit the school for any reason. On the other hand, such parents may be the most frequent of visitors, by the principal's invitation and as a result of abnormal behavior upon the part of the student. These parents are often embarrassed by their children's school status and do not care to come face to face with the teachers. Briefly, it is nearly impossible to establish any kind of rapport between these parents and teachers in an effort to form some kind of solution to the students' problems.

What then, can be done for the basic student and at the same time alleviate the teacher's dilemma and bewilderment? Modern mathematical teaching techniques are geared toward a sound preparation for future endeavor through teaching for understanding and appreciation of mathematical concepts. Most readers will probably agree that such is good. The basic student need not be deprived of such an approach. The key to his course of study lies in a modification of these efforts.

We must begin with something of a remedial nature which is genuinely interesting, preferably minus any remedial label attached to the text material or class. Unfortunately, such materials in the form of a fundamental text appropriate to the situation described in this writing, are nonexistent in the experience of this writer. We can, however, devise our own materials pertinent to our own situation. We shall attempt the task of making remedial materials and techniques more refreshing and interesting to the student.

Let us assume the worst and decide we must begin remedial work all the way back at counting. This time we shall hope to progress more rapidly and apply somewhat more mature techniques. If we are sincere in our efforts, we will now attempt things which are not ordinarily associated with the teaching of basic students. We shall be delightfully surprised to see them work to advantage if they are given an ample amount of time and devotion.

A small, four-place abacus with vertical wires which allow for counters (these can be made as a class project) can be of extreme value in teaching various counting processes. The writer would suggest beginning to count in some other number base after a brief warm up in the decimal system. The binary system is suggested because of its frequency of change from one place value to another. Simply describe for the students the function of the instrument and begin by having them note on paper *exactly* what they see in numerical form. For instance, if no counters are showing, the student would write 0 0 0 0. As the work progresses, the student should actually enjoy this new notation and attempt other number bases as well. Do not overlook the excellent opportunity to count backward upon the abacus as well as forward. Try some addition and subtraction in several number bases while using the abacus. Gradually wean the students to paper work alone. Students should make some discoveries about and gain insight into the "tenness" of our decimal system as a result of this work. It will be your decision as to the scope of your experiment, based upon the acceptance and general interest of the class.

Basic students enjoy constructions. The writer suggests that the class begin by constructing a square. The teacher should stress accuracy in both the construction and use of measuring devices. Careful attention should be paid to the quality of lines, that is, differentiating between construction lines and the final lines which make up the figure. The finer considerations will result in the students making many attempts. It may take as much as a week to get most of the students to present an acceptable square, but the efforts should prove to be rewarding.

Rather than be faced with a "so-what-now" attitude upon completion of the

square, the teacher should be ready to apply the figure to some practical problem. Here again, we shall use remedial work in a refreshing manner.

First, we shall alter the square slightly by including quadrants and diagonals, as shown in the example. Now let us suppose that we wish to find the product of 35×56.

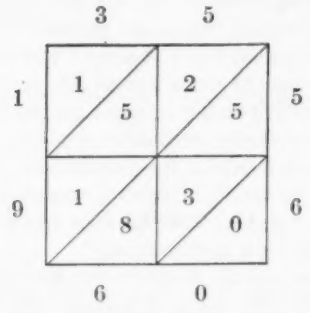


Figure 1

We can arrange the numerals around the square as indicated and write partial products with the tens in the upper portions of the small squares and the ones in the lower portions. The partial products will be added obliquely to give a final product which is read from upper left to lower right.[1]

Students will enjoy the departure from standard methods of multiplication and should be eager to attempt multiplying with three- or four-digit numbers in this fashion. Colored numerals in the final products will lend more interest to the project and provide additional satisfaction to the basic student.

The square may also be used for column and row additions, wherein the student may check his own work by sum agreements.[2]

[1] Harold D. Larsen, *Arithmetic For Colleges* (New York: Macmillan, 1955), p. 71.

[2] William G. Mehl, "Novel Drill Material for Signed Numbers," THE MATHEMATICS TEACHER, LII (April 1959), 247–249.

							X
	6	9	8	2	1	26	
	5	4	7	3	0	19	
	9	8	4	6	2	29	
	3	0	8	7	5	23	
	8	5	9	4	9	35	
Y	31	26	36	22	17	132	
							Z

Figure 2

Rows are totaled to column *X* and columns are totaled to row *Y*. Then *X* and *Y* are totaled to *Z*. Basic students thrive upon making up their own grids and exchanging them with other students. Any introduction of color here is also well to their liking.

The grid need not be restricted to the addition of integers. The same remedial techniques may be applied to the addition of fractions. See Figure 3.

Students should first find a common denominator for *all* of the denominators in the grid (preferably the L.C.D.) and then proceed as indicated in the example. Note that any necessary reducing is done in the final square at *Z*.

The examples shown in this article are intended to stimulate the creative and imaginative qualities of teachers. It is hoped that the material will be improved upon, as well as that it will provide suggested resources from which to begin anew.

Often we, as teachers, may tend to underestimate certain capabilities of students because of former achievement records. A poor record may have resulted from boredom, as is often true with gifted students. The basic student must also be challenged! Let us, then, do just that. Experiment. Try some signed numbers and bolster his ego by explaining to him that

				X
	$\frac{3}{10}$	$\frac{1}{8}$	$\frac{2}{5}$	$\frac{33}{40}$
L.C.D.	$\frac{12}{40}$	$\frac{5}{40}$	$\frac{16}{40}$	
	$\frac{7}{20}$	$\frac{3}{4}$	$\frac{5}{8}$	$\frac{69}{40}$
L.C.D.	$\frac{14}{40}$	$\frac{30}{40}$	$\frac{25}{40}$	
	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{7}{10}$	$\frac{80}{40}$
L.C.D.	$\frac{20}{40}$	$\frac{32}{40}$	$\frac{28}{40}$	
Y	$\frac{46}{40}$	$\frac{67}{40}$	$\frac{69}{40}$	$\frac{182}{40} = 4\frac{11}{20}$
				Z

Figure 3

this is what students of algebra study. (He is going to require the double dose of it anyway, if and when he attempts algebra.)

Introduce the basic student to primes and prime factoring. Show him how he can better perform operations upon fractions with this new knowledge.

Formulate some charts which can be dittoed, wherein the student can practice the additive method of subtraction and other elements of correct change making. See Figure 4.

Plotting co-ordinates on graph paper which produce "surprise" pictures proves exciting to the basic student.[3] Encourage the student to make use of color and his own creative ability to make graphs of his own to exchange with other students.

Do not overlook the possibilities which are to be found in solving cross-number puzzles.[4] Again, these may be created by students for a later exchange.

Encourage the basic student to attempt new (to him) ways of solving problems. Have him find sums horizontally. Write numbers and find sums, differences, and products in expanded form. Supply him with minuends and differences from which

[3] Margaret L. Carver, "Graphing Pictures," The Mathematics Teacher, LII (January 1959), 41–43.

[4] Louis Brandes, "Cross-Number Puzzles . . . ," The Mathematics Teacher, L (December 1957), 567–569.

Figure 4

Purchase	*Total*	*Tax*		*Total with tax*	*Amount paid*	*Change*					
		%	*$*			*.01*	*.05*	*.10*	*.25*	*.50*	*1.00*
\$2.35+\$4.99+\$.19+\$.15 +\$.35	\$ 8.03	2	\$ .16	\$ 8.19	\$ 10.00	1	1		1	1	1
\$6.14+\$.59+\$.98+\$.23+\$.10	\$ 8.04	3	\$ .24	\$ 8.28	\$ 10.00	2		2		1	1
\$1.88+\$2.69+\$.37+\$12.25	\$17.19	3½	\$ .60	\$17.79	\$ 20.00	1		2			2
\$3.33+\$4.95+\$17.75+\$11.95	\$37.98	2½	\$ .95	\$38.93	\$ 50.00	2	1				11
\$4.10+\$2.98+\$6.14+13.79	\$27.01	4	\$1.08	\$28.09	\$ 30.00	1	1	1	1	1	1
\$9.50+\$18.31+\$3.89+\$32.68	\$64.38	3¼	\$2.09	\$66.47	\$100.00	3				1	33
\$2.59+\$1.75+\$5.00+\$.49	\$ 9.83	2	\$ .20	\$10.03	\$ 20.03						10
\$.35+\$.55+\$1.38+\$12.55	\$14.83	2¼	\$ .33	\$15.16	\$ 16.00	4	1		1	1	

he may solve for the subtrahend. Likewise, find minuends when the subtrahend and difference are given.

Challenge the student's ability to follow directions, as well as his knowledge of the order of operations with directional exercises.[5]

[5] William G. Mehl, "Directional Exercises . . . ," *The Arithmetic Teacher*, V (April 1958), 152.

There are literally hundreds of different approaches to old ideas and methods. Experiment with the new and deviate from the old. Keep a folder or record of the items you have created and used. You will find that you are continually adding to your collection and will soon have compiled a "course" for the basic student of arithmetic.

Letter to the editor

Dear Editor:

Professor Salkind's letter on page 620 of the December, 1959 issue of THE MATHEMATICS TEACHER presents a very interesting proof of a theorem first discussed in the February, 1959 issue.

A rather elegant proof of the theorem (the difference of the squares of the distances of the vertex of the parabola $y=ax^2+bx+c$ from the origin and from either of its intersections with the x-axis is equal to the product of the abscissae of those intersections), and one that reveals its essential nature can be given thus:

Let O be the origin, A and B the intersections of the parabola with the x-axis, D the vertex of the parabola, and C the intersection of its axis of symmetry with the x-axis.

Then

$$\overline{OD}^2-\overline{AD}^2=\overline{OC}^2-\overline{AC}^2.$$

Now OB and OA give the roots of the equation $ax^2+bx+c=0$. Let

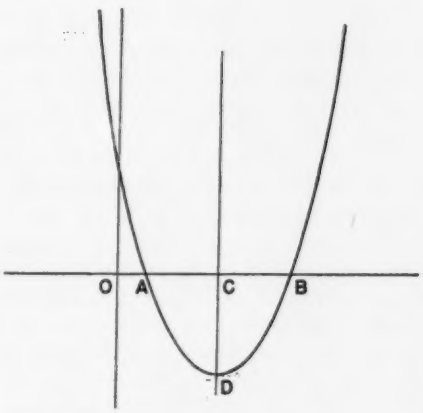
$$\overline{OA}=r_1, \quad \overline{OB}=r_2.$$

By symmetry

$$\overline{OC}=\tfrac{1}{2}(r_1+r_2);$$

and

$$\overline{AB}=r_1-r_2.$$



Therefore

$$\overline{AC}=\tfrac{1}{2}(r_1-r_2).$$

It follows that

$$\overline{OC}^2-\overline{AC}^2=\left(\frac{r_1+r_2}{2}\right)^2-\left(\frac{r-r_2}{2}\right)^2=r_1r_2.$$

Very truly yours,
DANIEL MALAMENT
Brooklyn, New York

Writing a number in different bases

ELBERT FULKERSON, *Southern Illinois University, Carbondale, Illinois.*

An example of changing the symbol for a number from one base to another base is presented here.

SEVERAL COMPARATIVELY RECENT texts dealing with the teaching of arithmetic present discussions on notation systems in different bases in order to emphasize the properties of the Hindu-Arabic system and to show how a thorough knowledge of these properties may contribute toward a better understanding of certain important arithmetical principles. Among other things, these discussions usually show how to convert a numeral from: (1) another base to base ten, (2) base ten to another base, and (3) a base other than ten to a base other than ten. The third type of conversion is generally performed by changing the numeral from the given base to base ten, and then by expressing this numeral in the desired base. A second and more direct method for making the conversion, however, is to change the numeral from the given base to the desired base without going through base ten. Inasmuch as this method requires an application of the addition and multiplication facts and algorisms in the desired bases, it is seldom discussed in connection with conversions because these are usually presented before performance of the fundamental processes in different bases is considered. It will therefore be the purpose of this article to show how conversions of the third type can be made by going through base ten and also made without going through that base.

Let it be required to express the number represented by 7236_{nine} in base five by going through base ten and without going through base ten.

SOLUTION BY GOING THROUGH BASE TEN

Write 7236_{nine} in polynomial form:

$$7236_{nine}=7(9)^3+2(9)^2+3(9)^1+6(9)^0$$

Now express the right member of this equation in base ten:

$$7236_{nine}=7(729)+2(81)+3(9)+6(1)$$

Next perform the indicated multiplications in base ten:

$$7236_{nine}=5103+162+27+6$$

Finally add in base ten:

$$7236_{nine}=5298_{ten}$$

It is now necessary to change 5298_{ten} to base five. This will be done by the method of repeated division.

$$\begin{array}{r|l} 5 & 5298 \\ \hline & 1059 \quad \text{R } 3 \leftarrow \text{1's} \\ 5 & 1059 \\ \hline & 211 \quad \text{R } 4 \leftarrow \text{5's} \\ 5 & 211 \\ \hline & 42 \quad \text{R } 1 \leftarrow \text{25's, or } (5\text{'s})^2 \\ 5 & 42 \\ \hline & 8 \quad \text{R } 2 \leftarrow \text{125's, or } (5\text{'s})^3 \\ 5 & 8 \\ \hline & -1 \quad \text{R } 3 \leftarrow \text{625's, or } (5\text{'s})^4 \\ 5 & 1 \\ \hline & 0 \quad \text{R } 1 \leftarrow \text{3125's, or } (5\text{'s})^5 \end{array}$$

The foregoing process can be rationalized as follows: By arranging 5298 ones into groups of 5 each, 1059 such groups are obtained with 3 ones remaining. Now, by arranging 1059 groups of 5 each into groups of 25 each, or 5^2, 211 groups of 25 are obtained with 4 groups of 5 each remaining. When 211 groups of 25 each are arranged into groups of 125 each, or 5^3, 42 of these groups are obtained with 1 group of 25 remaining. By arranging 42 groups of 125 each into groups of 625 each, or 5^4, 8 such groups are obtained and 2 groups of 125 each will remain. Finally, by arranging 8 groups of 625 each into groups of 3125, or 5^5, 1 group of 3125 will be obtained, and 3 groups of 625 each will remain. Therefore, the polynomial form for the number is $1(5)^5+3(5)^4+2(5)^3+1(5)^2+4(5)^1+3$, and the positional form is 132143_{five}. Hence, $7236_{nine}=5298_{ten}=132143_{five}$.

Without going through base ten

Again, write 7236_{nine} in polynomial form:

$$7236_{nine}=7(9)^3+2(9)^2+3(9)^1+6(9)^0$$

To convert the polynomial form into base five, first express the terms of the polynomial in that base. This requires that 7, 9, 2, 3, and 6 be each converted to base five. Inasmuch as each of these digits actually represents a collection of ones, then each can be expressed as collections of fives, that is in base five, by applying the simple process of counting. For example, to express 7 ones in base five, simply count seven things in groups of five. To illustrate, begin on the left in this collection of marks shown below and count by saying, "One, two, three, four, one group of five and zero ones over, one group of five and one one over, one group of five and two ones over." Thus, 7 ones can be expressed as 12_{five}. In like manner, 9 ones can be expressed as 14_{five}, and 6 ones as 11_{five}. Since no group of five can be obtained from the 2 ones or the 3 ones, these digits will not change when the polynomial is expressed in base five. Now by substituting the base-five equivalents for the original digits, the polynomial becomes:

$$7236_{nine}=12(14)^3+2(14)^2+3(14)^1+11$$

All terms of this polynomial are now expressed in base five. To expand and simplify the expression, it is necessary to multiply and add in that base. Inasmuch as the primary addition and multiplication facts for base five have not been committed to memory, the process of counting will be used to build up the following tables for these facts:

Addition facts: base five

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Multiplication facts: base five

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

Since the same algorisms which are used for performing the fundamental processes in base ten (the Hindu-Arabic system) can be used for performing these processes with numbers expressed in other bases, the above tables and the addition and multiplication algorisms will be used to find the needed powers of 14_{five}. To obtain the square of 14_{five}, multiply 14_{five} by 14_{five} as in Example A, and to obtain the cube of

1	2	3	4	5	6	7	8	9	10	11	12	13—Base ten
/	/	/	/	/	/	/	/	/	/	/	/	/
1	2	3	4	10	11	12	13	14	20	21	22	23—Base five

14_{five}, multiply the product found in Example A by 14_{five}, as in Example B.

Example A	*Example B*
14	311
14	14
121	2244
14	311
311	10404

To perform the multiplication in Example A, the multiplicand is first multiplied by 4. From the multiplication table in base five, it is found that $4(4)=31$. Write the 1 in the proper position and carry the 3. Next, $4(1)=4$, but to this it is necessary to add the 3 which was carried. From the addition table in base five, the sum of 4 and 3 is 12. By writing these digits in appropriate position, 121 is obtained as the first partial product. The second partial product is 1(14), or 14. Referring to the addition table in base five, the sum of the two partial products is found to be 311, which is the square of 14_{five}.

Example B indicates that the cube of 14_{five} is 10404, and was obtained by multiplying 311 by 14, using the same procedure as was used in Example A.

By substituting the needed powers of 14_{five} in the above polynomial, the following expression is obtained:

$$7236_{nine}=12(10404)+2(311)+3(14)+11$$

The right member of this equation is expressed in base five. To simplify this member, it is necessary to make the indicated multiplications and additions in this base, thus obtaining:

$$\begin{aligned}7236_{nine}&=130403_{five}+1122_{five}+102_{five}\\&\quad+11_{five}\\&=132143_{five}\end{aligned}$$

It will be noted that this is the same result as was obtained by going through base ten.

Have you read?

Ogilvy, C. Stanley. "Higher Mathematics at Lower Levels," *The Key Reporter*, Winter 1958–60, pp. 2–3, 7.

To appreciate the impact of change in mathematics, you should read this short article. The author points out how getting across the idea of "three" in a primitive society was considered high level mathematics. He goes on to state that in 1843 the mathematics program of Williams College consisted of algebra, Euclid measurement, surveying, navigation, spherical trigonometry, conic sections, and astronomy. The calculus was an elective for those who were very strong students. At Oberlin no calculus was given and at Princeton only one half year of calculus. What has happened in the past century? At least the first two years of college work is now taught in high school. Why? The pressures of more material available will push more and more material into the high schools and more of the high school mathematics will be in elementary school. Will algebra be given in grades five and six, calculus and complex variables in high schools? What will be deleted, and who is capable of deleting the nonessential material and teaching the essential? This author has some ideas you will want to weigh.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Sawyer, W. W. "Algebra in Grade Five," *The Arithmetic Teacher*, January 1960, pp. 25–27.

There are many ideas about where particular content should be placed in the curriculum. You will be interested in Mr. Sawyer's description of his work with algebra in grade five in Middletown Central School. His students were not only interested, but highly motivated. They not only got the "how" but also much of the "why." The nature of the process in algebra seemed to assist with a clearer understanding of mathematics. This article will interest you especially when you notice that the author hopes some of the people now in his class will become elementary teachers and not fear algebra in grade five.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mathematics teaching in secondary schools in England

A. P. ROLLETT, *Her Majesty's Inspector of Schools, London, England.*

A description of the mathematics program for secondary schools in England is presented here. English teachers generally use intuitive and graphic methods freely with young students. Logical treatment of a formal nature is deferred.

THE SCHOOLS OF ENGLAND fall into two main categories, namely those maintained by local educational authorities and those that are wholly or partly independent of aid from public funds. The organisation of the teaching of mathematics in the two categories is so radically different that each must be considered separately. There is much variety in both systems, and what follows is a simplified picture.

The maintained secondary schools draw their pupils from junior schools whose age range is 8 to 11 years. At about the age of 11 years the pupils are transferred either to a Grammar School or to a Modern School.[1] About one quarter[2] of the pupils, approximately those whose intelligence quotients are 110 or above, will go into the Grammar Schools or will do mathematics of the same standard in other types of school. With rare exceptions the pupils leave the junior schools with a knowledge only of the simpler processes of arithmetic and without having been taught even the most elementary notions of algebra or geometry. In consequence, the abler children reach the Grammar School after a period of mathematical stagnation, though the weaker pupils reach the Modern School with an imperfect grasp of the work. The two main faults, namely the failure to extend the abler children, and the prevalence of the teaching of processes without understanding, are due mainly to the inadequate preparation of the teachers in mathematics at the training colleges, though the syllabus of the examination taken at the age of 11 years is often allowed to have an undue influence. The main problem in connexion with new entrants to the Grammar School is that of getting quickly into gear and making up for lost time, and, in the case of the Modern School, that of restoring or creating confidence.

Every school in England is free to plan its own curriculum and, consequently, its own syllabus in mathematics. In spite of this freedom, the uniformity of aims and methods is surprising, unless it is realized that there are three influences at work in the direction of standardisation, or at least of the maintenance of standards. First: all Grammar Schools prepare their pupils for external examinations at the ages of 16 and 18 years, approximately. There are nine Examining Bodies, and a school makes its own choice; though the bodies maintain comparable standards, each has its distinctive "flavour." Second: the Mathematical Association, founded in 1870, is an active body, mainly of schoolmasters but with a strong minority of university mathematicians, which has pro-

[1] About 3 per cent of the pupils go on to comprehensive schools, but a quarter of these will do work of Grammar School standard in mathematics. About 3 per cent of the pupils go on to technical schools but these will do mathematics of the same standard as the Grammar Schools. There are also some hybrid schools.

[2] This is a national average; in fact the proportion varies from 10 per cent to 40 per cent in different areas.

mulgated through its regular and occasional publications a great deal of considered opinion on the best methods of teaching. Third: Her Majesty's Inspectorate includes a number of peripatetic mathematicians able to advise and encourage, and to help to disseminate the results of pedagogic experiments; they also organize courses for teachers.

The *Modern Schools* have existed in their present form only since 1945. The age-range is 11 to 15 years, but the fourth year in the school is broken by those who leave as soon as the permissive age is reached. In an increasing number of schools enough pupils remain for a fifth year to enable a course to be planned which leads to an external examination in mathematics. The present tendency, in a typical four-stream school, is to have the D forms entirely in the hands of form-teachers capable of dealing sympathetically with less able pupils; the C-forms are taught mathematics—little more than arithmetic and mensuration—by non-specialists; the A and B forms are taught by specialists, who give almost the whole of their time to mathematics. The A and B forms follow a course which would include elementary algebra, intuitive and practical geometry, numerical trigonometry, and perhaps a little mechanics. The work in many Modern Schools suffers through the absence of the support of regular homework. Progress is being made, through academic-year courses by secondment, and through improved standards in the Training Colleges, in increasing the supply of suitably qualified teachers. The mathematical potentialities of the abler pupils in the Modern Schools are realized in relatively few schools even now.

The commonest size of *Grammar School* has approximately 500 pupils; such a school would be organized in three *streams*, with an A, B, and C form at each stage. It is becoming increasingly common to have mathematics taught to forms 3A, 3B, and 3C, for example, at the same time, to allow regrouping into "sets"; an upper set which would forge ahead, a middle set of average ability (or two if staffing allows), and a lower set working to a modified syllabus. In England the belief is held that there is a wider range of aptitude for mathematics than for any other subject—except perhaps music, with which it has some affinity. This regrouping is not usually carried out until after the first year, but most schools discover that "the earlier the better." Such "setting" fails in its object if it is employed merely to make teaching easier by having more homogeneous groups and not to give each group its best treatment. Properly used the upper set should be at least a year ahead of the middle set at the end of the fourth form year. In too many schools, however, the work of the upper set is cramped through excessive caution in the fifth year, in view of the first external examination which is taken at the end of that year, and the lower set is given more than it can absorb. It is unfortunate that there are often two periods during which little progress is made—the transitions from junior school to Grammar School and from fifth form to sixth form. A pupil's development in mathematics suffers more from an interruption than in the case of most other subjects. About half the pupils leave at the age of 16 and the rest will stay for two years in the sixth form.

There are about twelve hundred maintained Grammar Schools.[3] The number of independent schools of similar character is much smaller, but there is one group of about two hundred, misleadingly known as *Public Schools*,[4] which is of great importance because its output of mathematicians is disproportionately large. These boys' schools, almost entirely residential, have an age-range of 13 to 18 years. They

[3] There are also 174 direct grant Grammar Schools, maintained only partly from public funds, which have the same age-range as the maintained Grammar Schools but in organisation incline to the independent public schools.

[4] This is an old usage, dating from the time when a boy might have a private tutor or be educated "in public"—with other boys.

are fed from small private Preparatory Schools whose age-range is 8 to 13 years. In these Preparatory Schools the boys begin algebra and geometry at the age of 9 years, so that when they enter the Public Schools, by way of a common examination, they have had a good start. They can be placed at once in the appropriate set since their attainment is known, and there is little interruption to their progress. The system of regrouping into sets of different character is more severe than in the maintained schools and the sets are smaller. In consequence the brakes are kept off and the abler boys make very great progress, soon reaching a point at which their work consists largely of private study. At the age of 16 years at least one third of the boys will be at least one year ahead of the national Grammar School average in mathematical attainment, and a much greater proportion will stay for a full sixth form course.

The view held generally among teachers of mathematics in England is that between the ages of 10 and 14 years most children—boys in particular—have an insatiable thirst for facts, ideas, and processes. In consequence it is felt that considerable progress can be made if the work is kept simple in character, and if intuition and graphic methods are freely used; while reasons adequate to the stage of the pupils' development are given, logical treatment of a formal nature is deferred. In the grammar schools and in at least the A-streams of the modern schools algebra can be started happily in the first year (pupils of 11 years), numerical trigonometry in the second or third year, followed quickly by logarithms, treated informally. Geometry begins intuitively in the first year and is gradually built up into a coherent system based on a large number of assumptions. The elementary notions of the calculus can be introduced even in the fourth year. It is believed that working knowledge of a new notion or entity is necessary before it is profitable to attempt to formulate a definition, and that postulational methods are out of place below the sixth form (pupils of 16 years and over).

The allowance of time for mathematics between the ages of 11 and 15 years inclusive is $3\frac{1}{2}$ to 4 hours weekly of instruction and about $1\frac{1}{2}$ hours of homework. In the sixth form the time varies enormously between one school and another; it may be as little as 7 hours weekly or as much as 15 hours, according to the amount of science taken. The school week may consist of 35 periods each of 45 minutes; in general a teacher will be in action for about 30 of these unless he is head of a department. There are no official textbooks.

It is convenient to employ the terminology of the Examining Bodies in indicating the content of the syllabuses in use at each stage. The examination papers taken at the age of 16 years (at the ordinary level) are of two kinds, Elementary Mathematics (taken by all boys and half of the girls) and Additional Mathematics (taken mainly by future specialists in mathematics or science). The papers taken at the age of 18 years (at the "advanced" level) are Advanced Mathematics (for those specialising in science) and Higher Mathematics (for those specialising in mathematics); there are also "scholarship" papers usually on the same syllabus which are designed to give the future mathematician an opportunity to display his aptitude.

It is a peculiarity of the teaching of mathematics in England that almost all work at the advanced level includes a substantial amount (up to half the available time) devoted to mechanics; the syllabus in Additional Mathematics also usually includes mechanics as an option. Elementary statistics forms part of the work in many sixth forms, often as an alternative to mechanics.

The syllabuses which follow are given in skeleton form as an indication of their content. The treatment expected can be inferred from the appropriate examination papers.

ORDINARY LEVEL

Syllabus A:

Arithmetic with practical applications.

Numerical Trigonometry.

Algebra: Formulae and their transformation; graphs; equations: simple, quadratic, linear, simultaneous, problems; negative and fractional indices; arithmetic and geometric series.

Geometry: the substance of Euclid, books I–VI.

Syllabus B:

Arithmetic and mensuration; formulae and equations; functionality; graphical study of simple functions; application of gradients to rates of increase, linear kinematics, maxima and minima; determination of a function from its gradient and applications of this to linear kinematics, areas, and volumes of revolution; geometry of triangles, parallelograms and circles (proofs of only the major theorems will be expected to be known); simple loci; the geometry of the simpler solids; sine and cosine rules, applications to geography and surveying.

[Syllabus B was introduced in 1945 and is slowly ousting Syllabus A.]

Additional mathematics:

Remainder theorem; logarithmic notation: the six trigonometric ratios and their relationships; simple trigonometric equations; circular measure; graphs of trigonometric functions. Differentiation and integration of powers of x, with applications to small increases, etc.; coordinate geometry of points and lines; conditions of parallelism and perpendicularity of lines; equation of tangent to a simple curve; equation of circle and tangent; loci. Moments; parallel forces; parallelogram and triangle of forces; composition and resolution of forces, velocities, and accelerations; equilibrium; centre of gravity; friction; rectilinear motion with constant acceleration; Newton's laws of motion; conservation of momentum; work and energy, conservation of energy; time of flight, greatest height, and horizontal range of a projectile; graphical methods.

ADVANCED LEVEL

Advanced mathematics:

Quadratic functions; finite series; permutations and combinations; binomial theorem, its use for fractional and negative indices, partial fractions; parabola, ellipse, hyperbola referred to rectangular axes; parameters; mensuration of tetrahedron, cone, and sphere; properties of the triangle, concurrencies, etc.; coaxal circles; inversion; trigonometry of small angles; addition formulae; general solution of trigonometric equations; inverse trigonometric functions; differentiation of trigonometric functions; exponential and logarithmic functions, their graphs and derivatives; use of exponential and logarithmic series; differentiation of products, quotients, functions of a function; integration by substitution and by parts; applications of the calculus to curves and to kinematics, etc.; equilibrium of one or more bodies under coplanar forces; smoothly jointed frameworks; mass centres; second moments; Hooke's Law; friction; relative velocity; mass, force, momentum, impulse, work, energy, power; direct and oblique impact; motion of two connected particles; projectiles; circular motion; conical pendulum; simple harmonic motion; motion on a vertical circle; motion of a body about a fixed axis.

Higher mathematics:

Infinite series; simple theory of equations; determinants, factorization, and multiplication; recurrence formulae; simple inequalities (A. M. v. G. M.); complex numbers, Argand diagram, de Moivre's theorem; hyperbolic functions; elementary projective geometry; involution; use of homogeneous and line coordinates; $S+kS'=O$; application of Taylor's and Maclaurin's theorems; length and curvature of curves; evolute and envelope; polar coordinates; implicit functions; partial differentiation, total differentials, change of variables; elementary differential equations; virtual work; the catenary; motion of a rigid body in a plane; conservation of angular momentum; more advanced problems.

"Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean."—*G. H. Hardy,* A Mathematician's Apology.

A new plan for instructing large classes in mathematics by television and films

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How can television be used most effectively in teaching a course in mathematics?

IN JANUARY OF 1957 the Mathematics Department of the University of Houston began preliminary study of an experiment designed to use television and films to attack the problem of teaching an increasing number of freshman mathematics students without a corresponding increase in faculty and physical facilities, while maintaining high academic standards. All known plans of instruction at other universities that utilized video in any form were investigated before a decision was made on the details of the proposed experiment. Plane trigonometry was selected as the subject to be taught because we thought that the subject matter was suitable for video and because of the large number of students who were enrolled in the course. This article discusses the details of the plan of instruction now being used in this experiment at the University of Houston. It also discusses some problems encountered in planning, developing, and executing the program along with a description of the plan for evaluating this method of instruction.

The principal features of the plan being used are listed below:

1. The course material was divided into twenty-seven lectures of forty-four minutes each. All lectures were carefully checked by members of the mathematics faculty and filmed by the University Film Center. Six members of the faculty participated as lecturers while others contributed as critics.
2. Lectures are presented to the students at the rate of two per week. Each film is shown twice over open circuit television (morning and evening) and twice by projectors in viewing rooms of the Audio Visual Center (afternoons and Saturday mornings). Thus a student is given an opportunity to view each lecture several times, if necessary, to master the subject matter. Viewing rooms are provided on campus for all televised lectures.
3. All students enrolled for credit are supplied with a television supplement which contains routine instructions for the course, a list of study aids, additional explanations, and *an incomplete set of notes on all lectures.* The student completes the notes in the supplement as he views the lectures.
4. All credit students are divided into sections of 25–36 students and each section is required to meet one hour per week with a member of the mathematics faculty at a regularly scheduled period. These conference sessions are used to discuss lectures that have been viewed by the students, to answer any questions arising from these lectures, to distribute and collect homework assignments, and to review for examinations.
5. Ten hours per week of voluntary help

sessions are scheduled in the afternoons and evenings with a student assistant in charge to answer questions about the solutions of specific problems.

6. The examination schedule includes two major examinations (midterm and final) and two minor examinations. The major examinations are two hours long and are given at an open period to the entire group. The minor examinations are one hour long and are given in the weekly conference sessions. Each major examination is preceded by a live televised review.

Many difficult problems arose in planning, developing, and executing this video plan of instruction. The first major problem was to design a procedure for producing a script for a lecture which was both acceptable mathematically and compatible with the over-all video program. The following procedure was agreed upon:

1. The lecturer was assigned a topic, with specified film footage. He then produced rough notes on this topic and presented them to the mathematics department for criticism.
2. The lecturer refined these rough notes in light of the criticisms and put them in production form.
3. A tape recording of the actual lecture, along with the notes in production form, was made available to the mathematics faculty for criticisms.
4. Final production form of notes was then prepared in light of the criticisms offered on the notes and on the tape recording.

To say that the notes were put in production form means that the notes were written in rectangles or "frames" containing seven to nine lines with a maximum of 20–25 characters on each line. The drawings, problems, reference formulas, etc., that were to be written on the board prior to filming were done in one color while the material to be written during the lecture was written in another color. Thus a critic could take a tape recording of the lecture and a set of notes in production form and simulate the actual film production.

One of the hardest problems to be solved was a personal one for the lecturer and one which differed with the temperament and psychological make-up of each individual involved. This problem was to adjust to the physical restrictions encountered during the filming process. While the lecture was being filmed and the lecturer was developing his subject matter he had to remember that the camera could view only a limited portion of the board, roughly 24×32 inches. These rectangular areas when marked off on the board were called frames. One row of frames was outlined on the upper part of the board and one row on the lower part of the board. The word "board" is used loosely, for the material was actually written on grey or blue building paper with black grease pencil or soft white chalk. The upper frames were used for drawings and reference formulas while the lower frames were used to prove theorems and to work problems. The lecturer had to remember to work within one frame and to signal the cameraman when he wished to move to another frame. This close co-ordination with the camera was necessary since the student sees only that portion of the board viewed by the camera. The lecturer also had to remember that he could write only 20–25 characters on a line of the frame, each character having a minimum height of $1\frac{1}{2}$ inches. He also had to remember to speak clearly and distinctly, to stay out of the line of sight between the camera and the board, and to go slowly enough for the student to complete the notes in his television supplement. Our experience showed that one had to practice in front of the camera until all of these restrictions could be adhered to by habit and reflex. Otherwise, when placed in this strange environment, the lecturer would usually forget a restriction which would make it necessary to stop production. In addition to remembering

all of these physical restrictions, the lecturer had to concentrate on what he was saying to prevent careless errors.

During the process of actually shooting the films, at least two mathematicians watched and listened for accidental errors. If a lecturer made an error and corrected it before the camera left that portion of the board, the error and the correction were allowed to remain in the film. If the error was not corrected by the lecturer, production was stopped and the scene was retaken.

The editing of the raw film stock for the master release print proved to be quite a problem because the film editor did not understand the language of mathematics and a mathematician was not available who could cut and edit films. Consequently, a mathematician and a film editor had to work together. Since a double system of sound was used in making the films, the task of editing and matching the picture with the sound was one which required trained personnel.

When a video type of instruction is substituted for the conventional classroom-teacher method, several problems immediately confront the student. First of all, taking notes on the lecture is much more difficult. If the student gets behind while taking notes in a conventional class, he can catch up by referring to the notes on the board as time permits. This is not possible with a video type of instruction. The student must keep up with the lecturer in taking notes because he has access to only the material in view of the camera at the moment. This problem was solved by the use of the television supplement which includes an incomplete set of notes on each lecture. All theorems, statements of problems, drawings, reference formulas, etc., which appeared on the board prior to the actual filming were reproduced in the television supplement. Each film was viewed before the corresponding portion of the television supplement was written and additional notes were supplied at any point in a lecture where the authors thought that the student would have difficulty in completing his notes.

Another problem which immediately confronts the student is that he cannot interrupt the lecturer for a question. This problem was partially solved by the lecturer in preparing his script. From his teaching experience he anticipated some of these questions and provided the answers. It was also partially solved by suggesting to the student that he write down each question in the margin of his television supplement and get it answered elsewhere, such as by viewing the lecture again, asking his instructor for an explanation in his conference session, getting the answer in a help session, or best of all, determining the answer himself by studying his notes and his text.

An additional problem for the student is his lack of self-discipline. All of his instructional opportunities are voluntary except his attendance in one conference session per week. An intelligent use of this freedom of choice creates quite a problem for the average student.

This plan of instruction deviates so radically from conventional classroom procedure that an intensive orientation program is necessary for students, counsellors, and conference session teachers in order to assure a smooth operation. The major part of the orientation is needed by the students. This problem was partially solved by distributing an orientation pamphlet to students, faculty, and counsellors at registration. The orientation of the students is also implemented by an orientation telecast at the beginning of each semester. Then too, a portion of the first conference session is devoted to orientation. This particular problem should decrease with time as more students become familiar with the plan of instruction and become less prejudiced toward the use of it.

Co-ordination of the entire program is a problem of major proportion. This involves informing many departments and

miscellaneous personnel of their particular parts in the program. This is accomplished by making a calendar of all events connected with the entire program and distributing the information to each person and department concerned. This method of disseminating information has many obvious weaknesses and requires regular checking of persons and departments affected.

Some of the advantages of this plan of instruction are:

1. Since the students are in class only one hour per week, a teacher is assigned two sections in lieu of one regular class. This saves 50 per cent of the teacher requirement for conventional classes.
2. Classroom requirements are only one-third of those for conventional classes.
3. It encourages the student to study and work independently.
4. It eliminates the "captive audience" status of the better students by allowing freedom to participate according to need.
5. The slower students may view repeat showings of the lectures.

The plan for evaluating this method of instruction involves the following studies:

1. A study of the problems which confront the students in using this plan of instruction.
2. A comparison of the subject matter learned by our students with that learned by students in comparable colleges and universities using the conventional classroom method.
3. A comparison of the performance in succeeding mathematics courses of students who studied plane trigonometry by the conventional classroom method with the performance of students who studied it by our video plan of instruction.

In order to accumulate data for the first study, a questionnaire was designed to explore all conceivable factors that might affect the student's learning experiences in the course. These data, along with the course grades, I.Q. test scores, and comprehensive reading test scores are being placed on data processing cards for analysis. No serious study of these data can be made until later. However, a preliminary observation of the I.Q. test scores, the comprehensive reading test scores, and the course letter grades of some 250 students clearly indicated that the letter grades from A through F were found over the complete range of I.Q. test scores and the comprehensive reading test scores.

The second study is complete. The Educational Testing Service Co-operative Plane Trigonometry Test was administered to our students at the end of each semester over a two-year period. The results follow:

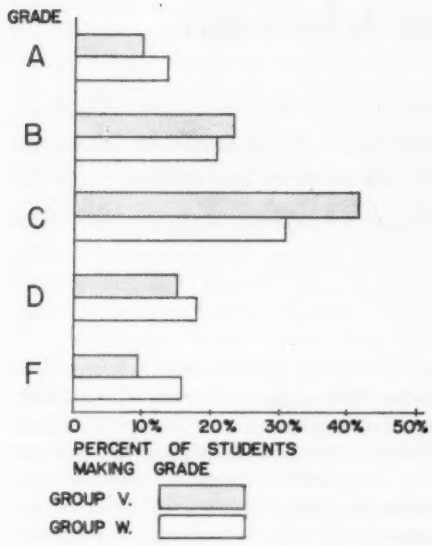
Class	Number of Students	Norm
Fall 1957	249	55.4
Spring 1958	253	56.9
Fall 1958	235	57.4
Spring 1959	207	57.2

The norm on this test for students in comparable colleges and universities is 57. The class of fall 1957 undoubtedly suffered considerably because of administrative difficulties encountered in the initial offering of the video program.

Data for the third study will need to be collected for at least two more years before any significant conclusions can be drawn. A preliminary study was made of data obtained from 188 students who completed a mathematics course which follows plane trigonometry. The results are shown in the following table and the graph on the next page:

Group	Number of Students	% A's	% B's	% C's	% D's	% F's
V	80	10	24	42	15	9
W	108	14	21	31	18	16

Group V studied plane trigonometry by our video plan of instruction. Group W studied plane trigonometry by the conventional classroom method.



It should be noted that Group W contained the students who came directly to the University of Houston from high schools where they had already taken plane trigonometry. Furthermore, these students had scores in the upper 10 per cent on their intelligence tests.

The question of whether or not a course can be taught by the use of television seems to us to be improperly phrased. The question should be, "How can television be used most effectively in teaching a course?" Our video plan of instruction would be improperly named "telecourse." Although television is the principal medium by which information is given to the student, the television supplement, the conference sessions, the repeat showings, and the help sessions all play an important part in teaching the subject to the student. This variety of instructional opportunities makes our video plan very effective for students having a great range of mental abilities. For example, the typical A and B students learn the course by viewing only one showing of each lecture and by attending only one conference session per week while the students having less aptitude, but sufficient desire, can learn the material by making use of the available instructional opportunities.

Our video plan of instruction is now beginning its third year at the University of Houston and it is generally conceded that it is doing a satisfactory job.

Letter to the editor

Dear Editor:

Your readers might be interested to know that the square to equilateral triangle dissection described by Chester Hawley in the February issue is one of the famous discoveries of the English puzzlist H. E. Dudeney. It appears as problem 26 in Dudeney's *The Canterbury Puzzles*. In his answer, Dudeney describes the construction but does not give a proof of its accuracy. I included the problem, together with Dudeney's construction, in my article on Dudeney in the *Scientific American*, June, 1958.

This in no way detracts from the value of Mr. Hawley's interesting article, but I think it might be worthwhile to correct the impression that the construction is due to Steinhaus, whose book simply reprints Dudeney's dissection.

Sincerely,
MARTIN GARDNER
Dobbs Ferry, New York

High school algebra via television

JOHN J. ANDREWS, *St. Louis University, St. Louis, Missouri.*

Mathematics has been and is being taught by television. In this study both teachers and students reported that they liked mathematics via television.

"CAN TELEVISION SERVE in the educational crisis facing the nation today?" is a question which more and more educators and laymen are asking. This report is one answer to the question. It is a case study of a mathematics course taught at KETC, Channel 9, to students in the metropolitan area of Greater St. Louis, which includes public, parochial, and private schools.

Two courses were offered during the year 1958–59, one in the fall—Intermediate High School Algebra, and one in the Spring—Advanced High School Algebra. Each course was sixteen weeks long, and telecast four days a week for a period of thirty minutes each day. Four-hundred sixty-four students, representing twenty-three schools, participated. At the end of the year, teachers, both classroom and television, and students agreed that courses via television had a definite place in school curricula and that, given a better understanding of what television education involves, television can be made an effective educational tool.

One cannot think of educational television without coming face to face with the problems of: subject matter, scheduling, methods of presentation (production, in television terms), qualification of television teachers, student-teacher acceptance, and of course, student achievement. Therefore, this report includes questions raised and answers gained in each of these areas.

The subject matter in these two courses covered all of the concepts called for in the syllabi of the leading school systems as well as that suggested by the Commission on Mathematics of the College Entrance Examination Board. During the summer the Intermediate Algebra syllabus was planned without the co-operation of the schools that were to participate in the fall, but for the spring term, the Advanced Algebra syllabus was determined through co-operation with participating schools. An outline of these two courses will be found at the end of this report. A basic text was adopted for *Intermediate Algebra*, but while this text was available in some schools it was not in others, so that subject matter and assignments were actually prepared for four texts in local use. This had several disadvantages. Homework problems could not be reviewed since all students did not have the same problems, and in texts other than the major one the assignments were not in sequence through the book, causing some confusion. To overcome these disadvantages the second semester, *Advanced Algebra* was produced with no text. Each day's assignment included the pertinent definitions and text as well as a minimum number of problems. Each class then used its own text as reference and source for additional problems. These minimum problems were then worked on the program the day after assignment. This divided every program into a review of previous assignment and current subject development. The secretarial department of KETC provided assignment sheets,

examination and supplementary material in sufficient quantity for every participating student. No effort or expense was spared in this preparation. Examinations were semi-objective, and a grading key was provided for uniform grading. Class results were submitted after every examination for statistical analysis. The results of this summary, with suggested grading was presented on the program prior to the next examination. A histogram of student scores, median, mode, and arithmetic mean were discussed briefly in this presentation. This brief introduction to statistics was culminated in the final week of Advanced Algebra with the discussion of standard deviation, normal curve, and related material.

Scheduling is a most serious problem for a large region with several independent school systems. A check chart of class schedules showed that a thirty-minute period from 1:55 to 2:25 P.M. had the greatest availability for the schools in this area. Due to overlapping of various schedules the thirty minute period was chosen as maximum for the program. Most schools had a twenty week semester, but vacation periods again conflicted so that both courses were limited to sixteen weeks. On alternate Fridays, examinations were given in the participating schools. On the intermediate Fridays opportunity was provided for personal instruction by classroom teachers and the visiting of these classes by the television teacher. During the year each class had the experience of personal discussion with the television teacher. Between fifteen and twenty schools were in actual participation with classes viewing regularly, while other schools used the program intermittently for special features, or for interested teachers and students.

The actual production of the programs required the co-operation of the entire studio. A script for each program was submitted two weeks in advance to provide time for the property department to prepare any props or stage settings. The first few programs were rehearsed with the director and camera men prior to the day of presentation, but after initial difficulties were overcome and general procedures decided on, a thirty-minute review of the script with the director on the day of the program was adequate unless there were some special stage settings. Two types of boards were used for presenting written material. One, a newsprint pad on an easel, and the other a roller board that exposed paper with an area proportioned to screen size that could be rolled forward and back if needed. A black wax pencil was used on the manila paper to give the appearance of black-on-white on the screen. This was used in preference to a blackboard and chalk. The stage setting normally consisted of the easel and the roller board. Homework was pre-transcribed and presented on the easel, and then a movement to the roller board for new material gave a slight diversion in the presentation. Some special props were used occasionally. A magnetic board and a felt board were available if desired. Several demonstrations were presented during the semester. When diagrams were presented with blocks and rings, proportionality was presented with letters on the magnetic board.

A few mathematical interest films of short duration (2 or 3 minutes) were shown when appropriate. (When discussing the ellipse, a film showing a satellite being put into orbit might be shown, etc.)

One technique of production, the use of tellops, deserves special comment. A tellop is a prepared card whose picture from a separate camera is superimposed on the major picture. All new and key words when they were introduced into a discussion were superimposed on the screen. As an example, when discussing the intersection of two sets, the word "intersection" would appear for about thirty seconds on the bottom of the board where the demonstration was being presented. This visual stimulus assisted in recognition of the word and its spelling. This technique was

used throughout the entire presentation. If several tellops are being used on the same program their sequence must be followed as planned on the script and gone over with the director. Two cameras were used and, when available, a Zumwald lens. This Zumwald lens is very useful in allowing the camera to enlarge any section of the board when needed. Only one problem, that of presenting lengthy statements, caused some concern. To overcome this difficulty, lengthy word problems and statements were included on the assignment sheet so that they could be read from these sheets during the program.

It is a little unusual for a teacher to say the same thing twice in the same way and many times the first way is best. For this reason in rehearsals one may only walk through, talk about, but not say what he plans to say on the program. The first appearance before the camera may be frightening, but a good director and a pleasant cameraman soon make one feel at ease. The use of magnetic tape and prerecorded programs tend to make the production of a series of programs less restricting than the daily live broadcast. Each semester there were several sessions at the studio when teachers of participating classes and KETC staff discussed technical problems and co-ordinated their activities.

The television teacher finds his job a challenge. Not thirty students, but hundreds, hear his words. This calls for extra care and preparation. This year I was the teacher, next year it may be you. The time in preparation for a half hour class is possibly three hours with timing and preparation of material. Knowledge of your subject is your greatest self-assurance and fear of the camera is soon replaced with pride of performance. A pleasing personality helps with the staff as well as with the students. Certainly here is adventure enough for anyone.

A summary of a questionnaire on likes, dislikes, good and bad features, completed after the course, gave this interesting information: Almost all schools liked the manner of presentation and printed materials. The best features indicated were: 1) all schools had the same course outline; 2) the television lesson required more attention on the part of the student; 3) more responsibility for learning was placed on the individual student; and 4) all students received identical instruction. The bad features were: 1) little or no personal contact between student and television teacher; and 2) questions could not be asked during the program. It was gratifying that both teachers and students reported that they liked "Mathematics via Television."

Student achievement is always a measure of teaching accomplishment. At the close of each semester a standard final examination was provided to determine the accomplishment. For Intermediate Algebra the Garman-Schrammel Algebra Test published by the Bureau of Educational Measurements, Kansas State Teachers College, Emporia, Kansas, was used. On this examination the median student score was at the 60th percentile. For a pilot class of 37 students, half freshman upper-track with no plane geometry and half average sophomores, the average was at this same level. This pilot class had no mathematics teacher other than the television teacher. They graded their own homework and discussed problems together under supervision. The Advanced Algebra class was given the Blyth Second Year Algebra Test, published by the World Book Company. Here the results showed a high degree of success. The median score was at the 90th percentile and 75 per cent of the students scored above the 74th percentile. Both classes showed a high degree of accomplishment. One might assume from this that the quality of student and instruction had improved in the second semester.

Mathematics has been and is being taught by television. The teacher as well as the participating students can be pleased with their part of this modern educational achievement.

COURSE OUTLINE
INTERMEDIATE HIGH SCHOOL ALGEBRA

September 8–11

The number system with a review of fundamental laws and operations. The set of natural numbers as a subset of the integers, as a subset of the rational numbers, as a subset of the real numbers, as a subset of the complex numbers. Polynomials and operations with them.

September 15–18

Equations conditional, identical, and inconsistent. Inequalities. Solution set. Solution of linear equations in one, two, and more variables, by graph and algebraic methods. Test over first and second weeks.

September 22–25

Verbal problems leading to equations in one, two, and more variables. Number, geometric, age investment, motion, lever, digit work, etc.

September 29–October 2

Type products, factoring, prime factors, H. C. F. and L. C. M. Test over third and fourth weeks.

October 6–9

Remainder theorem, rational roots, solution of literal linear equations in one or two variables. Introduction to fractional expressions.

October 13–16

Addition, subtraction, multiplication, and division of fractional expressions. Complex fractions. Test over fifth and sixth weeks.

October 20–23

Fractional equations in one and two variables. Verbal problems leading to fractional equations.

October 27–30

Functional relations, independent and dependent variables. Domain and range. Ratio, proportion, and variation. Graphical representation of linear equation in two variables with slope and intercepts. Test over seventh and eighth weeks.

November 3–6

Determinants. Properties of determinants. Cramer's rule. Independent, dependent, and inconsistent equations.

November 10–13

Laws of exponents, roots, radicals, fractional exponents, surds, and operations with radicals. Test over ninth and tenth weeks.

November 17–20

Solution of radical equations. Imaginary and complex numbers and operations with them.

November 24–28

No programs during this week. Comprehensive examination of first ten weeks. Special assignment.

December 1–4

Quadratic functions with maximum and minimum values. Solution of quadratic equations in one variable. Test over eleventh and twelfth weeks.

December 8–11

Nature of the roots of quadratic equations. Fractional quadratic equations and verbal problems leading to quadratic equations.

December 15–18

Systems of equations involving quadratic equations. Arithmetic and geometric progressions. Test over thirteenth and fourteenth weeks.

December 22–25

No programs during this week. Special assignment.

December 29–January 1

No programs during this week.

January 5–8

The logarithmic and exponential functions. Logarithms to base ten, use of tables, interpolation, antilogarithm. Fundamental laws of logarithms.

January 12–15

Computation with logarithms. Multiplication, division, roots, powers. Summary of semester's work. Test over fifteenth and sixteenth weeks.

January 19–22

No programs during this week. Comprehensive examination over semester's work.

COURSE OUTLINE
ADVANCED HIGH SCHOOL ALGEBRA

February 2–5

The postulates of a number field, introduction to the notation of and operation with sets.

February 9–13

Relations as subsets of a product space, functions of one variable with domain of variable and range of the function. Test over first and second weeks.

February 16–20

Linear equations, equivalent, conditional, identical, inconsistent. Type products and factoring.

February 23–26

Fractions and fractional equations. Verbal problems in one variable. Test over third and fourth weeks.

March 2–5

Linear equations in more than one variable and simultaneous solutions of linear systems.

March 9–12

Determinants, general properties, Cramer's rule, application to systems of linear equations to determine independence, dependence, or inconsistency of systems. Test over fifth and sixth weeks.

March 16–20

Laws of exponents, roots, radicals, fractional exponents, surds, and operation with radicals.

March 30–April 2

Quadratic functions, solution of quadratic equations in one variable. Verbal problems leading to quadratic equations. Test over seventh and eighth weeks.

April 6–9

Advanced topics in quadratics, nature of roots, equations quadratic in form.

April 13–16

Simultaneous quadratics, graphical, and algebraic solutions. Test over ninth and tenth weeks.

April 20–23

Variation, arithmetic, and geometric progressions.

April 27–30

Mathematical induction and the binomial theorem with applications. Test over eleventh and twelfth weeks.

May 4–7

Factor theorem, remainder theorem, synthetic division, rational roots.

May 11–14

Theory of equations, location, and approximation of irrational roots. Test over thirteenth and fourteenth weeks.

May 18–21

Permutations, combinations, introduction to probability.

May 25–28

Introduction to statistical analysis. Test over fifteenth and sixteenth weeks.

Letter to the editor

Dear Editor:

E. W. Gruhn's interesting article on Pythagorean triples in the December, 1959 issue of THE MATHEMATICS TEACHER ends with the conjecture that the pair of triples (3, 4; 5) and (7, 24; 25) comprises the only one exhibiting the property that an acute angle of the triangle associated with the second is double an acute angle of the triangle associated with the first. This is almost the exact opposite of the truth!

Theorem: Any Pythagorean triple determines a second triple having the required property. The second triple is primitive if and only if the first is primitive.

Proof: Let $(a^2-b^2,\ 2ab;\ a^2+b^2)$ be the first triple where $a>b$. Then the second triple is $(\pm[a^4-6a^2b^2+b^4],\ 4ab[a^2-b^2];\ [a^2+b^2]^2)$. The ambiguous sign merely insures that the first member of the triple is positive.

It is easily verified that these triples are Pythagorean. Also, if θ is an acute angle of the given triangle

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\frac{2ab}{a^2+b^2} = \frac{4ab(a^2-b^2)}{(a^2+b^2)^2}$$

which is precisely the sine of the angle opposite $4ab(a^2-b^2)$ in the second triangle.

For example, if $a=3$, $b=2$ we have the pair of triples (5, 12; 13) and (119, 120; 169). Or if $a=4$, $b=1$ we get the pair (15, 8; 17) and (161, 240; 289), etc., etc.

Further, if the given triple is primitive, then a^2-b^2, $2ab$ and a^2+b^2 have no common factor. In that event, neither can $4ab(a^2-b^2)$ and $(a^2+b^2)^2$ have a common factor; and conversely.

Very truly yours,
DANIEL MALAMENT
Brooklyn, New York

Edited by Howard Eves, University of Maine, Orono, Maine

John Napier and his logarithms

by C. B. Read, University of Wichita, Wichita, Kansas

It is doubtful that any teacher would today try to introduce the subject of logarithms other than by means of some use of exponents. It is, then, indeed hard to realize that Napier constructed his logarithms before the concept of exponents as we know them was developed.

John Napier was born in 1550, probably when his father was about sixteen years of age. He is known to have matriculated as a student at St. Andrews University, but records fail to show that he graduated. His first published work was *A Plaine Discovery of the Whole Revelation of St. John.* However, it appears that from an early age he was interested in mathematics as well as theology. He apparently did some early work in algebra and in arithmetic. Toward the end of the sixteenth century, Napier became perturbed at the fact that scientific progress was hindered by the great labor involved in numerical calculation. He deliberately attacked the problem of developing some means of lessening the work involved. He devised several mechanical aids, among these being sets of small rods often called "Napier's bones." In his approach it seems evident that Napier was (at least at first) interested in trigonometric computation, for he deals almost exclusively with sines.

The great contribution of Napier was the invention of what we now know as logarithms, although at first these were called by Napier "artificial numbers" or simply "artificials." The number corresponding to the logarithm was called a "natural number."

The exact date of the discovery, if indeed one can say there was a definite date of discovery or invention, is uncertain. Kepler tells us that in 1594 Tycho Brahe had information from a Scotch friend that there was a possibility of publication of the material. Actually, the computation of the table, or "canon" as the author called it, probably took several years. There are two books which are explanatory of the system, generally abbreviated as the *Constructio* and the *Descriptio.* Of the two, the *Mirifici Logarithmorum Canonis Constructio* is the most important, in fact the most important of all Napier's works.

The *Constructio* presents clearly his original conception of logarithms. The "canon," with instructions for its use, was published in 1614. The method of construction, although probably written several years earlier, was not published until 1619. It was published in Edinburgh, some two years after Napier's death (which may have been hastened by the strain involved in the development and computation of the "canon"). In the body of this book, the term "artificial number" is frequently but not exclusively used instead of "logarithm." The term "logarithm" is used in the title page, the headings, and in the ap-

pendix, which discusses the advantages of logarithms to the base ten.

This original work is relatively rare, few writers having had the privilege of examining a copy. In addition, the fact that the work is in Latin makes it less easy for a present-day student or teacher to follow, even if a copy were available. An English translation was published by W. R. Macdonald in 1889.

The book consists of sixty numbered paragraphs, occupying, in the English translation, less than fifty pages. (This does not include the appendix, which is entitled "On the construction of another and better kind of Logarithms, namely one in which the Logarithm of unity is 0," nor a supplement dealing with methods of solving spherical triangles with the use of logarithms.) The entire book needs to be studied to understand Napier's approach; a few necessarily brief comments may be helpful.

The first numbered paragraph defines a logarithmic table as " . . . a small table by the use of which we can obtain a knowledge of all geometrical dimensions and motions in space, by a very easy calculation."

As the treatment proceeds, the author finds it necessary to define arithmetical and geometrical progressions. Since accuracy is needed, he suggests taking large numbers for a basis, "but large numbers are most easily made from small by adding ciphers." He then says "these large numbers may again be made still larger by placing a period after the number and adding ciphers." It is found necessary to explain that 10000000.04 is the same as 10000000 4/100. It is perhaps not generally known that Napier was the first to use our present notation for decimal fractions.* It is not impossible that this was a by-product of his working out the invention of logarithms, although in another work Napier makes reference to the contributions of Simon Stevin. However, in spite of the simplicity of the notation proposed by Napier, it did not come into general use until long after his death.

Napier next presents a discussion of the accuracy of working with his artificial numbers—a discussion which has much resemblance to our present-day work on the accuracy of computation with approximate numbers.

To follow the text, it is necessary to understand that Napier's canon was not a table of logarithms of numbers as we know it (that is, of equally spaced numbers) but of "sines of arcs" for every minute from 0 to 90 degrees. It may be easier to understand Napier's approach if it is known that at this period the "sine of an arc" (which we call the sine of an angle) was the length of a line equal to the half-chord of a double central angle in a circle whose radius was unity. Hence Napier's explanations frequently represent both sines and logarithms by lines. Instead of calling the radius unity, he adds seven ciphers. From this radius he suggests subtracting its 10000000th part, obtaining 9999999; then from this he subtracts the 10000000th part, obtaining 9999998.0000001; again subtracting the 10000000th part he obtains 9999997.0000003. Continuing this process he creates, as he calls them, "a hundred proportionals." The "hundredth proportional" is 9999900.0004950. The resulting table is called the "First Table."

For his "Second Table" Napier points out the difficulty of forming fifty proportional numbers between the first and last numbers of his First Table (10000000.0000000 and 9999900.0004950). However, as he puts it, "A near and at the same time an easy proportion is 100000 to 99999," which will yield "sufficient exactness." So for the Second Table he starts by "adding six ciphers to radius and continually subtracting from each number its own 100000th part," obtaining fifty other proportional numbers, the last being 9995001.222927 (an unfortunate error in computation—the correct value being 9995001.224804).

* See, however, D. E. Smith, *History of Mathematics*, II (Boston: Ginn and Co., 1925), pp. 238 and 244.

The Third Table "consists of 69 columns, and in each column are placed 21 numbers, proceeding in the proportion which is easiest, and as near as possible to that subsisting between the first and last numbers of the Second Table." For ease in calculation, the proportion is considered to be 10000 to 9995.

Essentially these three tables constitute a table of sines, or natural numbers, "progressing in geometrical proportion" with no two numbers differing by more than unity (after rounding). Napier now proceeds to show how to place, beside the sines, or "natural numbers" (which decrease geometrically), their logarithms, or "artificial numbers" (which increase arithmetically). To do this he makes use of the concept of a "geometrically moving point" which, when approaching a fixed point, has its "velocities proportionate to its distances from the fixed point." With this concept we have essentially Napier's definition of a logarithm: "that number which has increased arithmetically with the same velocity throughout as that with which the radius began to decrease geometrically."

Having shown that "nothing is the logarithm of the radius," he discusses "the limits of the logarithm," which covers roughly what we would call the maximum error in a computed logarithm. He then develops certain laws relating to proportions, expressed in terms of logarithms. In modern notation, for example, if $a:b=b:c$ and we know the logarithms of any two of the three quantities, we may obtain the logarithm of the third; similarly for $a:b=c:d$. The explanation proceeds to show how to find logarithms of sines outside the limits of the table. There are numerous examples.

The above explanation seems quite involved. It may be of some help to note that in the First Table, the logarithm of 10000000 is zero; the logarithm of the next number 9999998.0000001 is 1; that of 9999997.0000003 is 3, and of 9999900.0004950 is 100. Rounding, for ease in computation, gives 100 as the logarithm of 9999900. Then, using a new ratio for the Second Table, more values can be inserted.

Napier then used existing tables of sines (disregarding decimal points which would appear in modern tables). He chose the number in his tables nearest the desired sine, and by use of "limits of its logarithm" (essentially interpolation) found the desired logarithm. Paragraph 43 demonstrates the method of finding the logarithm when the given sine is 9995000.000000. The nearest sine in the Second Table is 9995001.222927, from which he obtains, in modern notation: log 9995000.000000 = 5001.2485387.

Paragraph 58 shows that if the logarithm of "all arcs not less than 45 degrees are given, the logarithms of all less arcs are very easily obtained," thus paralleling our present-day tables which tabulate values to 45 degrees, then use the complementary angle. Paragraph 60 gives details for forming a logarithmic table—employing 45 pages, each capable of holding sixty lines of figures, and with seven columns of figures on each page. Once the pages are prepared, reference is made to paragraphs 49 and 50, which explain in detail how the calculations are performed.

Although it is very unlikely that anyone at the present time would wish to make up a "canon" of logarithms using Napier's method, there are several points of definite historic interest. First, with Napier's logarithms, as the natural numbers *decrease*, their logarithms *increase*. The logarithm of 10,000,000 is zero, the logarithm of 9,995,000 is 5,001.2, the logarithm of 9,900,000 is 100,503.3, the logarithm of 4,998,609 is 6,934,250, etc. Second, in the sense that we know it, Napier's logarithms actually had no base. Third, it is interesting to note that although decimal fractions were used in the construction of the canon, the logarithms given are large whole numbers—for example, paragraph 57 computes the logarithm of an arc of 34° 40′ as 5,642,242. (For brevity

Napier speaks of the logarithm of an arc rather than, as in modern usage, of the logarithm of the sine of an angle.) Finally, we note with interest that Napier's construction or demonstration was based on the concept of moving points, using what might be termed arithmetical and geometrical motion. Arithmetical motion implies constant velocity; Napier's geometrical motion involved variable velocity, continually decreasing.

By no means is it to be implied that credit should be withheld from Napier. On the other hand, when we read that Napier invented logarithms, it should be recognized that his canon, or table, differed in several important essentials from what we now recognize as a logarithmic table, and that his method of construction differs markedly from the explanation now given (as, for example, in a course in calculus) relative to the computation of logarithms. Finally, what we now often call natural or Naperian logarithms are not identical with those first developed by John Napier of Merchiston.

Naperian logarithms and natural logarithms

by Howard Eves, University of Maine, Orono, Maine

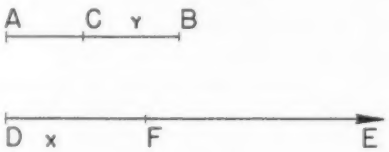
One frequently reads, in calculus textbooks, that logarithms having the base e are called *natural*, or *Naperian* (or *Napierian*), *logarithms*, leaving the impression with the reader that the logarithms devised by John Napier are the natural logarithms, or logarithms having the base e, and that Napier should be credited with the development of this system of logarithms. Actually, Naperian logarithms and natural logarithms are quite different, and it is the purpose of this note to emphasize the difference by obtaining an expression for the logarithm function of Napier in terms of the familiar logarithm function of today. We shall denote the Naperian logarithm of y by Nap log y, the familiar logarithm of y to base b by $\log_b y$, and the natural logarithm of y (the special case where $b=e$) by ln y.

As we know, the power of logarithms as a computing device lies in the fact that by them multiplication and division are reduced to the simpler operations of addition and subtraction. A forerunner of this idea is apparent in the formula

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)],$$

newly discovered in Napier's time, and it could be that Napier's line of thought started with this formula. In any event, Napier at first restricted his logarithms to those of the sines of angles.

Napier labored at least twenty years upon his theory, and, whatever the genesis of his idea, his final definition of a logarithm is as follows: Consider a line segment AB and an infinite ray DE, as shown in the accompanying figure. Let points C and F start moving simultaneously from A and D, respectively, along these lines, with the same initial rate. Suppose C



moves with a velocity always numerically equal to the distance CB, and that F moves with a uniform velocity. Then (ac-

cording to Napier) DF is the logarithm of CB. That is, setting $DF=x$ and $CB=y$,

$$x=\text{Nap log } y.$$

To avoid the nuisance of fractions, Napier took the length of AB as 10^7, for the best tables of sines available to him extended to seven places. Then we have $AC=10^7-y$, whence

$$\text{velocity of } C=-dy/dt=y.$$

That is, $dy/y=-dt$, or integrating, $\ln y=-t+C$. Evaluating the constant of integration by substituting $t=0$, we find that $C=\ln 10^7$, whence

$$\ln y=-t+\ln 10^7.$$

Now

$$\text{velocity of } F=dx/dt=10^7,$$

so that $x=10^7t$. Therefore

$$\begin{aligned}\text{Nap log } y=x&=10^7t\\&=10^7(\ln 10^7-\ln y)\\&=10^7\ln(10^7/y)\\&=10^7\log_{1/e}(y/10^7),\end{aligned}$$

and we have an expression for the Naperian logarithm in terms of the familiar log function (with base $1/e$).

It follows that the frequently made statement that Naperian logarithms are natural logarithms is without basis.

Have you read?

Rapoport, Anatol. "The Language of Science," *E.T.C.—A Review of General Semantics*, Summer 1959, pp. 445–458.

The subtitle of this interesting article is "Its Simplicity, Beauty and Humor." Most people feel that the vocabulary of science is composed of long words, difficult to spell and impossible to pronounce, or they think of it as prosaic words used in factual reports Read this article and you will be certain neither of these concepts is true. The author points out that mathematics and physics have a rich language, yet the vocabulary is small and consists of short, common words. Symmetry of expression, figures of speech, and rich imagery abound in this language of science. Key terms in mathematics such as group, ring, ideal, field, trace, norm, limit, converge, function, perfect, compact, ordered, and neighborhood are simple and meaningful. How do these meanings develop? One starts with the common and compounds their meanings to more complicated notions. The language of science uses simple words to denote exceedingly complex ideas. The language of science is not prosaic, witness such poetic terms as oscillating curve, frequency, feedback and even the J-th particle. You will want to read this, even the jokes.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Segman, Ralph. "Pounds or Grams," *Science News Letter*, August 15, 1959, pp. 106–107.

It has been said that only what you can measure can you know. After reading this article you will wonder if such is the case. Did you know there are more than eighty-five different standard weights listed by the Bureau of Standards, that there are two different sized pounds, three different quarts, three different miles, and two different tons? Do you know about the gill, perch, chain, furlong, link, angstrom, and all the rest? I suppose you are aware we still use avoirdupois, troy, and apothecaries' weights in a variety of places while ninety per cent of the world is using the metric system, advocated by Washington and Jefferson, and made legal in 1866. Watch for the bill in Congress requesting that the Bureau of Standards make a report on the feasibility of transferring to the metric system. It can be done. In 1955, Eli Lilly, a large pharmaceutical company, started a conversion program and now uses the metric system completely. This company has over 2500 items and 10,000 employees, but in less than five years all measurements are made in the more efficient metric system. Get your students to read this article. They are the ones who will be making the decisions.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

An intramural mathematics league

by Harold Westby, Sibley Junior High School, West St. Paul, Minnesota

If you are looking for a new way to promote a mathematics club or, more important, increased achievement in mathematics, try an intramural league with homeroom mathematics teams competing against each other. An intramural mathematics league has been highly successful at my school.

At Sibley Junior High School, West St. Paul, many attempts have been made to maintain participation in mathematics clubs. These clubs met after school, and because of this only a few members stayed for the meetings. At most only about a dozen members could be expected to participate in the best club activities. As a result of this poor attendance, the eighth-grade mathematics club started looking for ways to increase its membership and attendance. The club decided to sponsor an eighth-grade intramural mathematics league. Each member agreed to publicize the organization of this new league. Students went to the eight homerooms in the eighth grade and asked that each homeroom select five members to form a mathematics team. All eight homerooms responded enthusiastically and elected teams. A seven-game schedule was then set up so that each homeroom competed with each of the other homerooms in a round robin. At the end of these games, a champion homeroom team was named and a pennant awarded to this team.

Each contest consisted in two teams solving unique mathematical problems like the samples below. The teams were given twenty minutes to solve a set of ten to fifteen problems. At the end of each contest, the problems were corrected. Each individual as well as each team was given one point for each correct solution. The top paper usually had about nine out of twelve problems solved correctly. In view of the limited time and the difficulty of the problems this indicated intense effort on the part of the competing teams.

With five students from each homeroom participating as a team, the contest now had forty students taking an active part. It was expected that many of these students would continue in the mathematics club after the seven-week contest was over. Each week the team standings were posted and the students showed a great deal of interest in the standing of each

team, especially in their own homeroom team. Individual totals were also posted weekly, and this created even more interest since prizes were awarded to the individual with the highest standing. The top five individual winners received recognition at the local awards program in the spring.

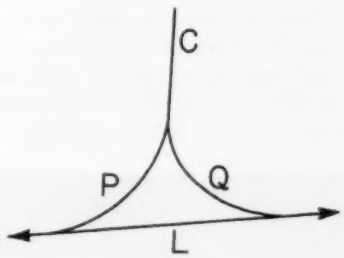
Some of the sample problems from these contests are given below. The simpler examples are needed for the less able team members and the more challenging problems for the top students. A larger number of challenging problems than easy problems are needed, since it is the top students in each homeroom who become the members of homeroom teams. In fact, all our eighth graders with the highest achievement records were in the list of the top point-getters. It is interesting to note that even the most difficult problems were solved by at least one student.

We think the intramural mathematics league has been an excellent stimulant to better achievement in mathematics at our school. At the same time, it has improved our homeroom program. Try it in your school and you may be pleasantly surprised with the results.

SAMPLE CONTEST PROBLEMS

1. In the number 35,678,243,100, how much greater is the value of the 3 on the left in comparison to the 3 on the right?
2. If you have a pair of dice each with only four faces and the number 1, 2, 3, and 4 on the faces in the usual manner, what is the possibility of getting a total of 8 if you toss them at random?
3. Arrange four 4's to get 3, for example $\frac{(4\cdot 4)-4}{4}$.

   Now find another way of arranging the four 4's to give 3.
4. Find the average of these numbers: 26, 42, −7, 36, and −18.
5. Fill in the missing numbers in this series: 989988987986.
6. Three men play a game with the understanding that the loser is to double the money of each of the other two. After three games, each has lost just once and each ends with $24. With how much did each player start?
7. While three watchmen were guarding an orchard, a thief slipped in and stole some apples. On his way out, the thief met the three watchmen one after the other. To each in turn he gave half of the apples he then had, and two besides. Thus he managed to escape with one apple. How many had the thief stolen originally?
8. A railroad track and two spurs form a triangle, as in the figure. The portion of the track at C is just long enough to accommodate one caboose, but it will not take a locomotive. Initially there is a locomotive on the open track, a caboose at P and a caboose at Q. How can the positions of the cabooses be interchanged?



9. How could you arrange five half-dollars so that each touches the other four?
10. Two pedestrians walk along the same road in the same direction. The first, walking at 4 miles an hour, starts out 8 miles in advance of the second, who walks at 6 miles an hour. As they start, a dog belonging to one man leaves his master and sets off for the other man at 15 miles an hour. As soon as he reaches the second man, the dog returns at once to his master. The dog continues to run back and forth until the second man overtakes the first. How far does the dog travel?

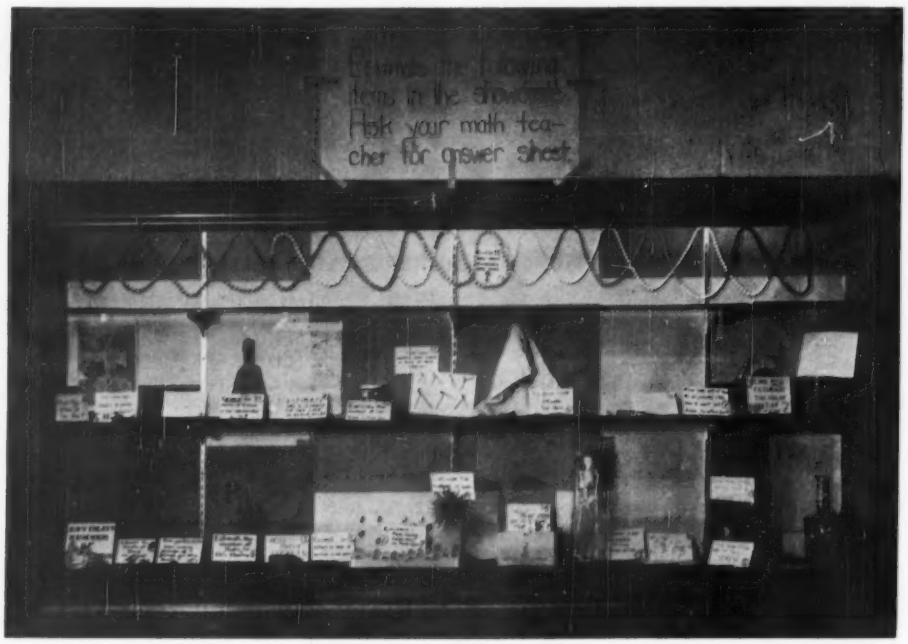
An exhibit on estimation

by Jack J. Wickert, Junior High School, Kalamazoo, Michigan

Estimating reasonable answers and solutions to problems is accepted by most mathematics teachers as a desirable goal of instruction. But proficiency in estimation is not easily attained through typical textbook problems. While discussing a display for our main showcase to be prepared by an advanced junior high arithmetic class, a suggestion was made for a display in which estimating would be the theme. During a class period spent discussing the idea, many interesting and workable suggestions came from the students. Each pupil was assigned the task of obtaining one item to be displayed that would involve estimation. In each case, something of unusual interest was sought. For instance, when it was suggested that a piece of chain be used and that it would be necessary to estimate the number of links, one member of the class used a girl's pendant necklace. It was more eye catching than a common chain. Again, when the number of words on a printed sheet was considered, a comic book page was displayed rather than some less interesting page of a textbook.

Beside being of value mathematically, the showcase was an eye-catching display that stimulated a great deal of discussion. Care was taken to have each item properly labeled so that it would be of interest to the viewer. For several days, whenever students were free, a crowd always gathered around the display.

The following are some of the twenty-



eight items displayed in the exhibit case:

A small planter in the shape of two hands containing pennies. How many pennies?
A girl's necklace. How many links?
A comic book page. How many words?
A jar of olives. How many olives?
A sheet of paper with x's typed in a disorganized arrangement. How many x's?
Five drawn angles. Estimate the size of each angle.
A piece of material with random array of gold stars in the pattern. How many stars?
A design composed of colored pop beads. How many beads?
A whisk broom. How many straws?
A plastic skeleton 1 foot high. What is the weight of the skeleton?
A large box of chalk with some sticks missing. How many pieces of chalk?
A five cent bag of peanuts. How many peanut halves?
A partial reel of magnetic tape. How many feet of tape?
A single ice-cube form. How long will it take for a cube made with this form to melt at room temperature?
A styrofoam mobile with toothpicks stuck into it. How many toothpicks?

Because of the great amount of interest in the exhibit, the student council voted to award prizes to the two students whose estimations were most nearly accurate. To encourage participation, sheets with spaces for each answer were distributed through arithmetic classes. Students had a week in which to check the showcase and make their estimations, then the completed sheets were returned to the class which set up the display. Here the answers were checked against the true count. Up until now no one had known the exact amount in each display so that everyone had an opportunity to participate. The award of the prizes to the contestants who were the most nearly accurate was an exciting climax to a worthwhile class project.

I

Computercade

An unusual educational program is being conducted in Arizona high schools. Inaugurated by the University of Arizona, the program's purpose is to convince Arizona high school students of the importance of mathematics in daily life and to show them how study of the subject can be important to their future careers.

Called a "Computercade," the UA program features a van-mounted $50,000 electronic I.B.M. calculator, which is touring the state's high schools to show students how these electronic machines work. In many border cities, the show has been attended by Mexican children.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

A Concrete Approach to Abstract Algebra, W. W. Sawyer (San Francisco: W. H. Freeman & Co., 1959). Paper, 233 pp., $1.25.

When you buy a book you do not, ordinarily, consider on which shelf in your bookcase you will later put it. You may have a little trouble in classifying W. W. Sawyer's new book *A Concrete Approach to Abstract Algebra*. It is about algebra—that much is clear. Although originally written for high school teachers studying mathematics at a National Science Foundation Institute, it is not a textbook in the usual style. On the contrary, Sawyer offers a rather informal discussion of the formal characteristics of modern algebra. At the same time, the book does contain "Questions" and "Exercises" for the student, with answers provided on the final pages, so that it can play the role of a textbook in suitable situations, such as institutes and some types of college courses which have similar aims. The book contains nothing explicit about methods of teaching, but a perceptive teacher can learn much about his art by studying not only the content itself, but also how the author has selected, organized, and presented it.

Sawyer wants, first of all, to communicate and develop the viewpoint of abstract algebra. Attention is focused upon the structure of various algebraic systems. The term "concrete" in the title has to be interpreted broadly. It is true that the author sometimes introduces structural ideas by describing some relatively simple calculating machines, electrical circuits, and other concrete objectives which constitute a realization of the structure he has in mind. Arithmetic is abstract to an elementary school pupil, but becomes relatively concrete to the high school student who is studying a good algebra course. In similar fashion, high school algebra serves as a relatively concrete basis for the study of other systems in a program of more advanced study. It is these extended senses of the word "concrete" that apply to the approach of this book.

Teachers and others who would like to take a tour of some of the high spots of modern algebra with Sawyer as an engaging guide should not be repelled by the possibly unfamiliar names of the places they will visit. Here are a few of the eleven chapter titles: "Arithmetics and Polynomials," "Finite Arithmetics," "Linear Dependence and Vector Spaces," "Vectors over a Field," "Trisection of an Angle." Those who have been over the same ground before will find the going easy. Neophytes will have to study the guide's book with care. He assumes the reader is "reasonably familiar with elementary algebra," but the reviewer believes a background this limited would be insufficient for more than a superficial view of the terrain. Nevertheless, Sawyer's way of leading into new territory, his skill in constructing examples to serve as steppingstones, and his candor in exposing not only his own tactics in conducting the tour, but also corners that others too often leave obscure, go far toward compensating for mathematical immaturity. For instance, take his discussion of the distinction between the *variable* x and the *indeterminate* x (pp. 61–70). Similarly, the statement of his point of view toward the distinction between the natural number 4 and the fraction (4, 1) or 4/1 along with a variety of similar situations (pp. 108–114) should help unsophisticated readers see what some of their colleagues are arguing about.

In the last fifty or sixty years a number of books with similar purposes have become available. Most of these have tended to be global in their coverage of mathematical content and points of view. In contrast, Sawyer has restricted himself to an exposition of algebraic ideas with the formal aspects dominant. With this limitation in scope, he has gone to extraordinary lengths to help the reader understand what the game is all about.

One of the most attractive features of this book is its low price. In these days of high prices for books of all kinds, and especially for technical books, the average reader should have no difficulty in getting his money's worth from this one, even if he doesn't get all the way through it. However, this low price entails some sacrifices. There is no index, there are a few misprints, and the paper (except for the cover) is of low quality.

If the book is to be used as a textbook for class instruction, let the teacher beware. He had better "know his stuff" rather thoroughly, for he may not be able to handle a class using this book in his customary way. When Sawyer wrote it, he put the material in the hands of his students, and "did not in the classroom try to expound the same material again. . . . In the classroom we simply argued about this material. Questions were asked, obscure points were clarified." This method of teaching is dangerous for one who is not steeped in the subject matter. On the other hand, students in more conventional courses will surely find the book useful as an auxiliary textbook. It can substitute in part for the "bull sessions" of students through which much of their understanding of mathematics has customarily been acquired.—*Maurice L. Hartung, University of Chicago, Chicago, Illinois.*

Introduction to Mathematical Thinking, Friedrich Waismann (New York: Harper and Brothers, 1959). Paper, x+260 pp., $1.40.

The subtitle of this book is *The Formulation of Concepts in Modern Mathematics*. Whether this book actually merits such a subtitle depends upon what "Modern Mathematics" means to the reader. If the structure of the systems of numbers is at the heart of modern mathematics, then this exposition of mathematical ideas certainly merits its subtitle. Proceeding from the natural numbers, the author constructs the integers, the rational numbers, and eventually the irrational numbers. These extensions are very carefully done with the definitions listed initially and followed by the traditional type of theorems and proofs found in texts dealing with the real numbers. Since this book is not intended to be a textbook, the number of theorems and proofs is limited, and there are no exercises, although the proofs of some theorems are left to the reader.

It would be a mistake to regard this book only as a development of the number system. The book is much more than that. In many instances the author delves into the philosophy of mathematics, particularly in pointing out contradictions, criticisms, and alternate routes of development. For instance, in the integers he notes two extensions of the number domain which yield two entirely distinct number systems. One is the traditional ordering of the integers (Figure 1), and the second is that of Wallis (Figure 2) in which the path from the positive numbers to the negatives includes ∞ rather than 0.

Figure 1

Figure 2

Two developments of the irrational numbers are given, Cantor's theory and Dedekind's theory.

An especially strong point for this book is the author's manner of stating problems. Before proceeding into a rigorous construction, the author poses questions which give the development a logical direction and enable the reader to do some serious thinking before the solution to the problem is given. A weakness is the arrangement of topics. The natural numbers are used as the building blocks for the integers and rational numbers, but the foundation of the arithmetic of the natural numbers follows the construction of the integers and rationals.

In view of the increasing emphasis in secondary mathematics on the structure of mathematics, this book seems to be a wise investment for the teacher of secondary mathematics. Some of the concepts treated in the latter part of the book are relatively difficult and will require considerable time and serious thought on the part of the reader.—*Mack L. Whitaker, Florida State University, Tallahassee, Florida.*

Mathematics and the Physical World, Morris Kline (New York: Thomas Y. Crowell Company, 1959). Cloth, ix+482 pp., $6.00.

This book differs from most books on mathematics or the physical sciences because it stresses the role mathematical reasoning plays in the study of nature. The author compares and contrasts the mathematician's and the scientist's methods. He shows the parts played by deduction, induction, analogy, and trained intuition in the discovery of new mathematical and scientific knowledge. Problems originating in everyday life situations are used to illustrate how mathematical reasoning can prevent man from reaching quick, erroneous conclusions. The reader also learns how, as knowledge of mathematics and science increased, man overcame his fears and superstitions. The book is, however, mainly the story of how one scientific or mathematical theory has led to another or to the subsumption under it of explanations for other phenomena. This history covers the period from the time of the Pythagoreans (about 550 B.C.) to the present day. The mathematical content varies from simple arithmetic to differential equations, and among the scientific topics discussed are astronomy, light, sound, gravitation, projectiles, and electromagnetism, to mention but a few.

The book has several praiseworthy characteristics. The author's style of writing makes reading easy and interesting for the reader who otherwise might be frightened away by too technical a presentation. An occasional phrase or anecdote gives evidence of a pervading sense of humor. Moreover, one senses the author's enthusiasm for his subject. The emphasis upon the relationships among mathematical concepts and between mathematical and physical concepts—the leading of the reader from one generalization to others at higher and higher levels—makes the reader keenly aware of unity and organization in the universe. Thus understanding is increased. The reader also finds illustrations of the conciseness and power of mathematical symbolism. The brief sketches of the lives of great mathematicians and scientists which are woven into the exposition add interest and often serve to illustrate the driving power of man's curiosity.

On the other hand, the author from time to time departs from precise mathematical language. For instance, axioms are sometimes not carefully worded. Moreover, when reading about the possibility of leaving this solar system, one finds the clause "so that he just reaches infinity." Often, however, such lapses are followed

by more precise explanations. Some typesetting and other errors should be corrected.

The book, however, is well written and will prove well worth reading. Teachers can find some implications for the teaching of mathematics and science at all grade levels. At least parts of the book should have motivational value for the junior or senior high school student or the university student. It is also recommended to the adult wishing a quick review of the topics discussed, and to any adult curious about his physical environment. Brief discussions of such modern problems as "With what velocity must man leave this earth if he wants to reach the moon?" "If he wants to leave this solar system?" and "What would happen if a man stepped into a hole dug right through the solid earth to a point on the opposite side?" will catch the fancy of many a reader.—*Velma I. Rust, Carleton University, Ottawa, Canada.*

Selections From Modern Abstract Algebra, Richard V. Andree (New York: Henry Holt and Company, 1958). Cloth, xii+212 pp., $6.50.

This text contains a sampling of topics in modern algebra and is an outgrowth of lecture notes "used for a two-hour sophomore-level course in abstract algebra at the University of Oklahoma" taken by mathematics majors concurrently with calculus, and portions used "at summer institutes for high-school mathematics teachers" and as "enrichment material with a group of advanced high-school students."

Chapter 1 contains the postulates for an integral domain motivated by considerations of properties of the integers, some elementary number theory, and a short discussion of the nature of proof. Equivalence relations and integral congruences provide the material of Chapter 2, which is followed by a chapter on binary Boolean algebra where the development proceeds primarily in terms of isomorphic electrical networks. Chapter 4, entitled "Groups," leads to the concept of quotient group and terminates with a statement of the Jordan-Hölder theorem. Chapters 5, 6, 7, and 9 are concerned with matrices and the related topics of systems of linear equations and determinants The postulates for a ring and a field are given, and ideals and residue class rings are defined in Chapter 8.

Throughout the body of the text, as well as in the lists of exercises, sequences of questions are set up to induce the student to make conjectures. Even one in sympathy with this technique of presentation may find it overemphasized. The frequency of the unanswered "whys" may leave the reader with the feeling that space should be provided, as in a workbook, for answers.

The mention of situations where algebraic concepts find use has great merit when done judiciously. However, when it is overly frequent, as in this text, with little or no indication of how the use occurs, it is questionable whether anything is accomplished other than impressing the student with the vastness of mathematics and its application.

Finally, on the manner of presentation, many students would find it an affront to their personal sense of dignity to be faced with such admonishments as "Did you, yourself, prove that the statement of Example 4 was false before reading the last paragraph? If so, congratulations. If not, ask yourself 'Why not?' Was it because you were too lazy to work it out for yourself with the answer so handy?"

A deficiency in content is the absence of a unified presentation of basic set theory and notation, despite Chapter 3 on Boolean algebra. No real precision is given to such terms as "subset," "contains," "ordered pair," and "set equality." Concerning set notation, the relation "$\subset$" is only implicitly described on page 1, while "$\in$" is defined in a footnote in Chapter 4.

Some definitions are incomplete in their formulations. It is not clear, because of the informality of approach, whether this is intentional. For example, a subgroup of a group G is defined simply as "a set of elements of G which, by themselves, form a group," which is particularly puzzling in view of the fact that the author is very explicit in stating that a well-defined operation and an equivalence relation are necessary to the definition of group. Occasionally, similar incomplete formulations of theorems occur. It would seem that the mathematically naïve reader particularly needs formulations which are mathematically sound and not open to erroneous interpretations.

A most interesting and commendable feature of the book is that many of the exercises are suggested readings in such periodicals as the *American Mathematical Monthly, Scripta Mathematica, American Scientist,* and *Scientific Monthly.* From these the student should certainly obtain an awareness that mathematics is a dynamic subject and that current mathematical research and activity can be as exciting as any other scientific work.—*Jewell E. Schubert, University of Illinois, Urbana, Illinois.*

Trigonometry, Dorothy Rees and Paul K. Rees (Englewood Cliffs, New Jersey: Prentice-Hall, 1959). Cloth, xi+318 pp., $3.96.

This trigonometry is certainly novel when compared to many others specifically intended for high school in that "general angle" is initially used as domain of definition for the trigonometric functions. This is in agreement with arguments of those who have criticized high school trigonometries in that, generally, definitions are first restricted to acute angles.

This approach is apparently in line with the authors' attempt "to maintain a proper balance between analytical and numerical work" along with their claim to "lean somewhat toward the analytical side" (Preface, p. vii). Whether the authors succeed is problematical, although surely the intent merits applause. But certainly the obvious advantages in de-

velopment afforded by beginning with general angle are, in the reviewer's opinion, far from fully employed. For example, the addition formulae for sine and cosine are still developed geometrically for acute angles only (Chapter 8).

All the traditional topics are covered, with most of them treated in conventional fashion. From the pedagogical point of view, the content seems, at times, to be handled in more adult fashion than usual for a high school text. The explanations are often brief. The editorial format is cleanly done, with easy-to-read page layouts and a sufficient, unencumbered use of figures. The book is completely done in black and white, a relief (at least to the reviewer) from the currently popular but sometimes absurd use of color. Exercises are sufficiently frequent and numerous. Editorial errors seem to be at a minimum. (But the reviewer would suggest that Figures 1-6 and 1-7 should have a specific reference in the text, that Figure 11-3 should be labeled '$\theta = \arctan u$'; that remarks on p. 96 concerning Table I should be referred to pp. 236–257.)

Sometimes the explanations seem to proceed too speedily, e.g., more attention to Cartesian co-ordinates, variable, and function would seem in order in Chapter 1; the treatments of periodic function, reduction formulae, and inverse function seem too quick. This reviewer also feels that the ratio of drill-type to discovery-type exercises is too large.

Those who consider careful mathematical language important in high school mathematics instruction will find the usual errors in this text. No distinction is made between number and numeral. The function concept is not developed carefully, nor is it in the spirit either of Dirichlet or of the more modern approaches being advocated by several national groups. Hence, the concepts of function and function value are confused. The reviewer feels there are some inconsistencies, e.g., between the definition of identity (p. 49) and of equation (p. 173); between (properly) leaving tan 90° undefined (p. 18) and giving the range of tan θ, for θ from 45° to 90°, as 1 to ∞, with the symbol "∞" left unexplained (p. 81). But, in all fairness, these errors are neither more frequent nor more serious than those in most similar texts.

In conclusion, if the reader is looking for a conventional high school trigonometry with initial definitions in terms of general angle and with an attempt to emphasize the analytical, he should study this text carefully. But if he is looking for a text to meet the recommendations of current national groups, then the reviewer suggests that this text does not fill the bill.—*Robert Kalin, Florida State University, Tallahassee, Florida.*

BOOKLET

Math Fun, Ray Skotte and Yngve Magnuson (Minneapolis: Ray Skotte and Yngve Magnuson, 6380 Monroe Street, Minneapolis 21, Minnesota, 1959). Paper, 88 pp.

My first impressions of *Math Fun damentals* were hard to clarify for I thought it crude, yet it seemed fresh in thought content.

I took it to school. My shop mathematics boys were curious about the book. It was off my desk going the rounds all that day. Some of the boys offered to fill in the spaces. It was shown to the teachers of an independent girls' school. The younger teachers liked the book; they used words like "cute," "clever," and "different" in describing it. The older teachers in the school said they wanted no part of it; it was not neatly presented. A few of them did admit that the problems were good ones, new and different. A headmaster said the cover was repulsive to him. One of the men in our high school, said "Ideas are fresh and interesting. It ought to be fun to work. Certainly it offers a lot of practice with practicality. The only complaint is that it is crudely done." One good student looked at it carefully and said, "I think that book would get better results because it awakens your interest."

The book could be used for enrichment material below junior high school level and for remedial practice in a practical way in junior or senior high school. Pupils who have shied away from regular mathematics might enjoy working in this way.

The book contains page after page of mimeographed charts and pictures in red, black, and blue ink. Some are clever and neat, others very crude.

Examples of "sets" are evident in the lessons on book sales. One should find the "null set" in the list of names of librarians who purchased the technical books: example, Ipsqueak and Lisayso.

Practical problems with charts and tables to fill in do encourage more mental arithmetic and less paper and pencil dependency. This is very good for the bright youngsters The older, slow learners would like to tabulate their answers and throw away scratch papers which disclose crutches employed to find answers.

The idea of substitution seems to be paramount in the authors' minds in presenting the material. Opportunities for using plus signs were left out on pages 24–25. Eventually XYZ must be taught as a product. Why use XYZ as a sum, when $X+Y+Z$ would show the full meaning? In the next lesson omission of the dash between $A-C$ would have taught the reading of a line properly in a natural way.

The lesson on Easy Term Payments is a good practical lesson, and the Missionary Percentage problems are especially worthwhile and meaningful.

The summary on the back cover of the paper bound book makes it easy to find any certain kind of manipulative skill problems.

In the final analysis the reviewer finds she must cast her vote with the younger teachers. She is sure that many teachers and parents would find places to use this workbook to advantage.—*Virginia J. Gibbs, Central High School, Columbus, Ohio.*

DEVICES

String models of the quartic, intersecting cylinders, hyperbolic paraboloid, and right helicoids, Educational Division of Industrial Research Laboratories, Inc., P. O. Box 471, Hempstead, New York. $10.00 to $18.75 each.

Mathematical models have long been recognized as an effective aid in the teaching of three-dimensional geometry, and certain figures which are generated by the motion of a straight line in three-space lend themselves to representation by "string models," with each string occupying one position of the moving line. The simplest such figures—e.g., planes, cylinders, cones—are readily visualized by most students, but the more complicated surfaces, such as the hyperboloids, hyperbolic paraboloids, etc., are more difficult for many students to "see" and to draw. For these students, especially, string models of such figures are most helpful, since they illustrate not only the various positions of the generating line but also the types of plane cross section which may be obtained from each solid.

The models available from Industrial Research Laboratories include many which would be unsuitable for most high school mathematics classes—various helicoids and quartic solids, for example—but also include a few which could be used by classes studying the graphing and equations of the conic sections, the intersections of these curves, and (for the most advanced students) volumes and areas enclosed between intersecting quadric surfaces or conic sections. The most useful models for these purposes would be the following:

1. truncated and two-napped cones, with movable planes, showing how the change in position of the plane affects the cross section;
2. hyperbolic paraboloid, showing very clearly the saddle point and the various parabolic and hyperbolic cross sections; and
3. two cylinders of equal radius, intersecting at right angles to form the famous solid whose volume is of interest in calculus.

The reviewers did not see the models in the first group, but feel they would be the most helpful of all those listed, since they could at least be used to illustrate something studied by almost any high school mathematics class. The hyperbolic paraboloid, although it could be used most effectively only in solid analytic geometry and calculus, is a very good model of its kind for students who have studied the graphs of parabolas and hyperbolas, and who are learning to draw three-dimensional figures. The intersecting cylinders would, it was felt, be more effective if the strings were heavier or if there were more strings: in one advanced class, only two students out of twenty were able to see the right-angled cross sections well enough to include them in a sketch of the figure.

The framework for all the models is clear plexiglass, giving them a fine appearance as well as a high durability. These are models which could be passed around a class without fear of damage, as long as students were careful about handling the elastic threads. Threads of different colors are used to trace the paths of different generating elements—thus two colors are required both for the hyperbolic paraboloid and for the intersecting cylinders. Models may be purchased either already assembled, or unassembled with directions which can be followed by at least some students. Prices of the models vary from $10.00 for an unassembled quartic solid to $18.75 for the assembled hyperbolic paraboloid, with a difference of $2–$4 between an unassembled and an assembled model of a particular solid. Although this seems rather expensive from the standpoint of materials alone it is to be remembered that these figures cannot be mass-produced, and that to construct a mathematically accurate figure (as these are said to be) requires several hours to drill holes and to complete an accurate frame for stringing. (Anyone who has tried to make string models will probably agree that one is fortunate indeed if one does not spoil several parts before completing one acceptable model!)—*Marie Bauer and Mildred Brannon, Urbana High School, Urbana, Illinois.*

"As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

"But when these sciences joined company, they drew from each other fresh vitality and thenceforward marched on at a rapid pace toward perfection."—*Lagrange.*

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Infinite, finite, infinity

by John N. Esposito, East High School, Knoxville, Tennessee

If a teacher in high school leads his students to be inquisitive and critical he will sooner or later have to face the questions, "What do you mean by infinite? What is infinity?" I encourage my students to ask questions and to try to come to some conclusions of their own. I find that these students are not afraid to rush in where angels fear to tread. They like to try to think in the abstract. They are eager to learn; therefore at such times I try to build to partial answers which will be satisfactory and consistent with future work.

To treat the mathematical ideas of infinity I try to build from first-year algebra to second-year algebra. From the first year of algebra I have built on the question, "What is the largest number you can write?" The classroom discussion produces some interesting ideas. Some of the things the students and I will discover together are a googol, a googolplex, and Sir Arthur Eddington's number, 136.2^{256}, which he gave as the number of protons in the universe. What is the number of grains of sand on all the beaches of the world? What is the number of all the leaves on all the trees in the world? I invite them to think of collections that have large numbers of things in them. We incorporate the words *infinite* and *finite* in our discussion at this time without making any definitions, but trying to build concepts and understandings. I hope to bring the students to thinking about collections of things—all the grains of sand on all the beaches, all the people in our classroom. The students are told that these are sets. A set is just a collection of things or elements.

We have been talking about physical things—grains of sand, chairs, books. All of these could be counted if we had enough time. We found in our discussions that no matter what number was chosen, a larger one and a smaller one could be chosen. Then we may ask about the set of all positive integers (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, . . .). How much time would it take to write down all of them? In this way I hope to build the first concepts of infinite and finite. In second-year algebra, if not sooner, the question some student will usually ask is, "How can we tell whether a set is finite or infinite?" Once the question is asked, I may start toward a definition.

It seems to me that a teacher must now go back in history and explain the development of this concept for the students. The students may be intrigued by the Zeno puzzles: it is impossible to cover any given distance, Achilles and the tortoise, and the arrow in flight. One can make up some excellent examples to fit these stories. The point I wish to bring out is that Zeno probably thought the sum of an infinite geometric series was infinite.

I usually start my explanation with the sum of the series, ($\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$ $\frac{1}{16}$ $\frac{1}{32}$ $\frac{1}{64}$ . . .). How can I find the sum since I cannot even find the last number? This may be just how the mathematicians of history thought about it. I want to reach an agreement among the students on the meaning of infinite and finite sets.

I attack the subject using the ideas of sets and one-to-one correspondence. I have previously introduced the idea of one-to-one correspondence to my students. The only new concepts I need to introduce are subset and proper subset. If every member of set B is a member of set A, then set B is said to be a subset of A. For example, the set of all the girls in the room is a subset of all the people in the room. Many examples can be given. I hope that some student will discover that the definition allows any set to be called a subset of itself.

If B is a subset of A and at least one member of A is not a member of B, then B is a proper subset of A. All the girls in the room is a proper subset of the set of all the people in the room. I ask the students to give examples of subset and proper subset. When they have a fundamental understanding of this concept I can bring in the one-to-one correspondence.

We consider the positive integers again and the set of positive even integers.

No matter what number I pick in set A or in set B the students can find a larger one; but they will agree that every member of set B is a member of set A. In addition, there is at least one member of set A that is not in set B; therefore we have a proper subset. The idea that we can put set B in a one-to-one correspondence with set A may take a little time; but the students I have tried this with have amazed me with their ability to understand this and other ideas I have introduced. I am of the opinion that it is wise to talk up to the students and assume that it is possible for them to absorb and understand more than the teacher may think.

The definition now follows. If a set can be put in one-to-one correspondence with a proper subset of itself, then the set is infinite; if this cannot be done, then the set is finite. I do not mention explicitly that any proper subset which can be put in one-to-one correspondence with an infinite set is itself infinite. If a student notices this, it makes me very happy.

I now spend some time going over this concept. The above is what I mean when I say the set ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, . . .) is an infinite set. I can show how I may take a proper subset of this set and put it in a one-to-one correspondence with the original set. We call this an infinite geometric series. This does not contradict anything students have learned before. I show them that in this world all the sets of physical things are finite. The sets of objects the teacher used in grade school and all the other examples they were given there dealt with finite sets; therefore the part could never be put in one-to-one correspondence with the whole. All the grains of sand and all the rain drops are finite sets. Only with numbers, ideas—imagination, if you wish—can I produce an infinite set.

$$\begin{array}{cccccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8 & \cdots & n & \cdots & \text{set } A \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & & \\ 2, & 4, & 6, & 8, & 10, & 12, & 14, & 16 & \cdots & 2n & \cdots & \text{set } B \end{array}$$

I must make sure that the students do not confuse infinite and infinity. Often at this point we discuss infinity. We consider what happens to the fraction $1/X$ as X approaches 0, which we write $1/X$ as $X \to 0$. We substitute some values for X, $X=5$, $X=2$, $X=1$, $X=1/2$, $X=1/4$, $X=1/10$, $X=1/100$, $X=1/1000$ etc. The value of $1/X$ changes from 1/5, to 1/2, to 1, to 2, to 4, . . . to 1000, etc. The students agree that the answer is getting larger. Now I talk about the number 1/100,000. They agree that this is a small number. Can I choose a number smaller, such as 1/1,000,000? Of course the answer will be "Yes." By this process I hope to show that, regardless of how small a positive number I consider, I can always choose one smaller. As we choose smaller and smaller values of X, the successive values of $1/X$ get larger and larger. We can say that $1/X$ increases indefinitely as X approaches 0, or we often say that $1/X$ approaches infinity as X approaches 0. We are speaking of the function $1/X$ as $X \to 0$. I do not try to discuss the absolute value of $1/X$ or speak of X approaching 0 from the right or from the left. Using the symbol in place of the word infinity and remembering that 1/0 is meaningless, then $1/0 \neq \infty$. I avoid the concept of a place at infinity or any geometric interpretation of such a place. If necessary I tell the students to wait until a later date or development in their studies.

I tell the students that I am leaving many things unsaid, but I will try to see that my definitions and those of their future work in mathematics will not conflict, although it will be necessary in the future to extend and elaborate on these meanings and these definitions.

"Learning is often greatly influenced by the student's awareness of his need for what is to be learned, or of its potential usefulness and value to him. One of the major responsibilities of the teacher is to help students formulate well-defined goals, and to encourage students to resolve to reach these goals—in short, to help them adopt appropriate purposes."—*Maurice L. Hartung* in the Twenty-First Yearbook of the National Council of Teachers of Mathematics.

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Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D.C.

NCTM convention dates

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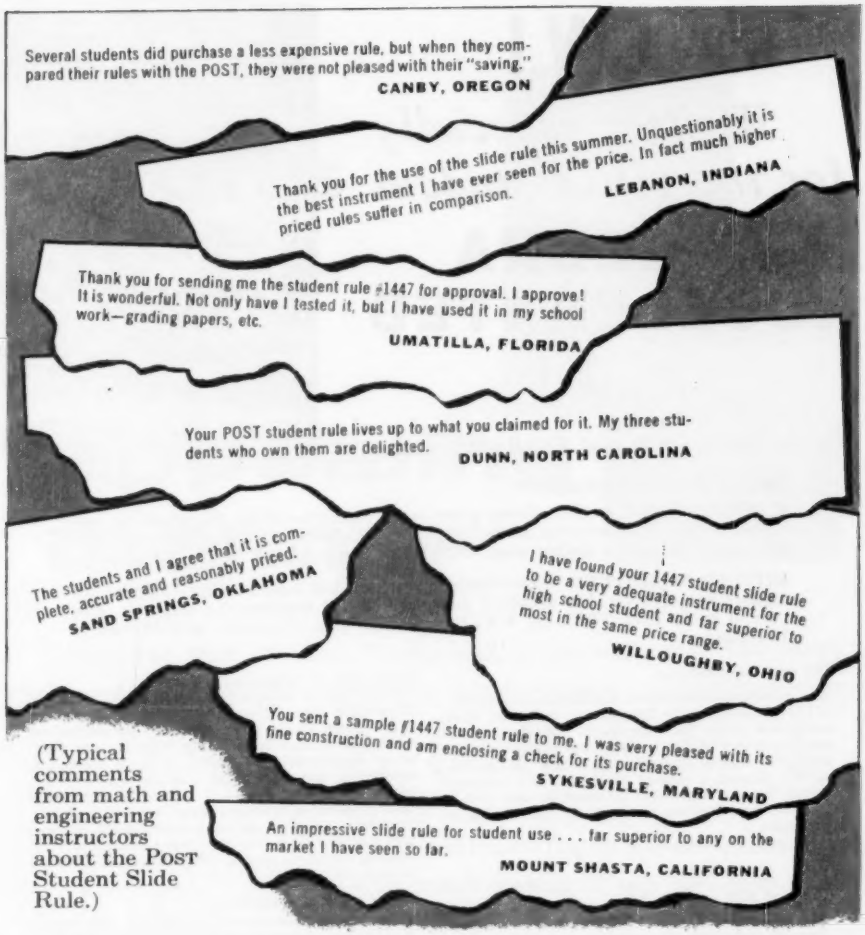
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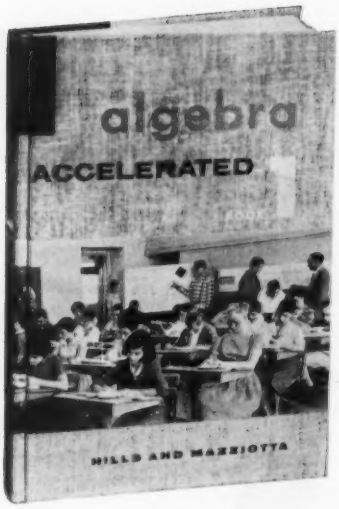

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E. JUSTIN HILLS, author of four other books on mathematics, has served on two California state mathematics committees and two seminars at U.C.L.A. Junior College and high school teacher since 1924, Ph.D., University of Southern California.

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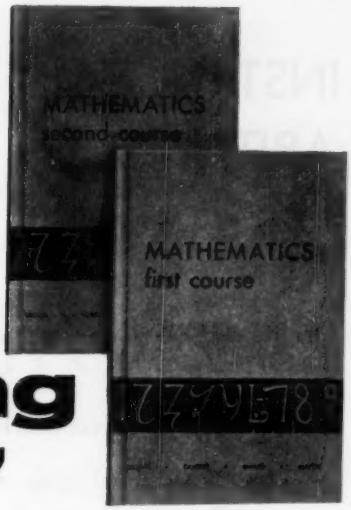

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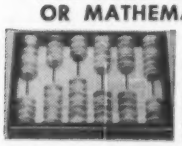

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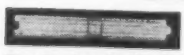

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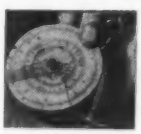

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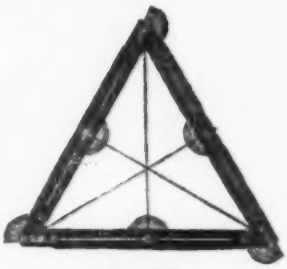
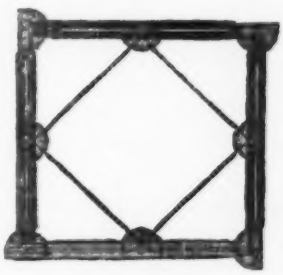
Dynamic figures permit observance of continuity and locus concepts as the lengths of sides, interior lines, locations of points, and angles are changed. A great variety of figures may be produced and measurements made of angles and lengths of sides. From these measurements, conclusions may be drawn which are the theorems of geometry—the inductive approach.

The instruments are accurately made of bright-colored, anodized aluminum and will last for years even with hard student use. Their cost is only a few cents per student per year.



No. 7500
Extensible Triangle



No. 7510
Extensible Quadrilateral

No. 7545
Universal Circle

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Schacht Manual
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Quantity prices apply for complete-class installations.

Write for a copy of the Welch Mathematics Instruments and Supplies Catalog.

The Schacht Devices meet the specifications of the CCSSO Purchase Guide No. 1745.

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